Correlated Pseudorandomness

Achieving faster secure computation through pseudorandom correlation generators

Geoffroy Couteau



Université
de Paris

Start of the Story: Circa 2016

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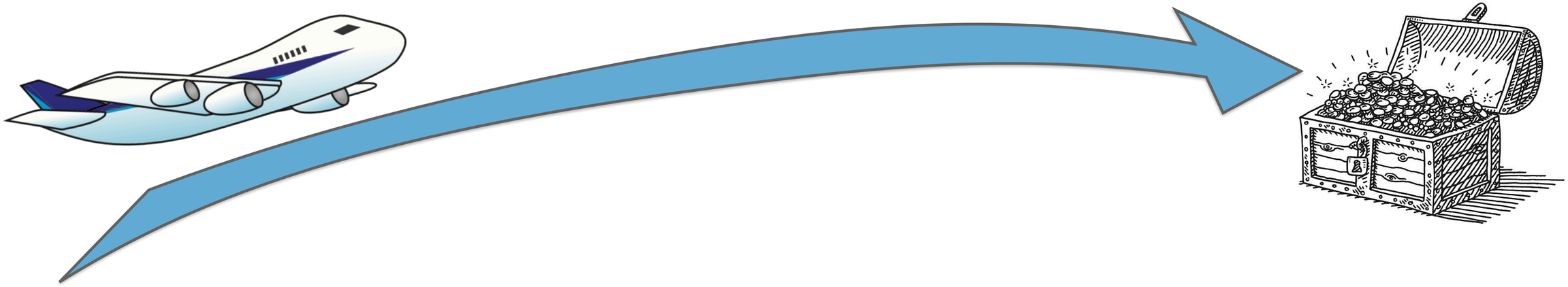
A few months and a CCS’17 paper later, we concluded that the answer was ‘not so much’


7 years later

The Journey through Correlated Pseudorandomess



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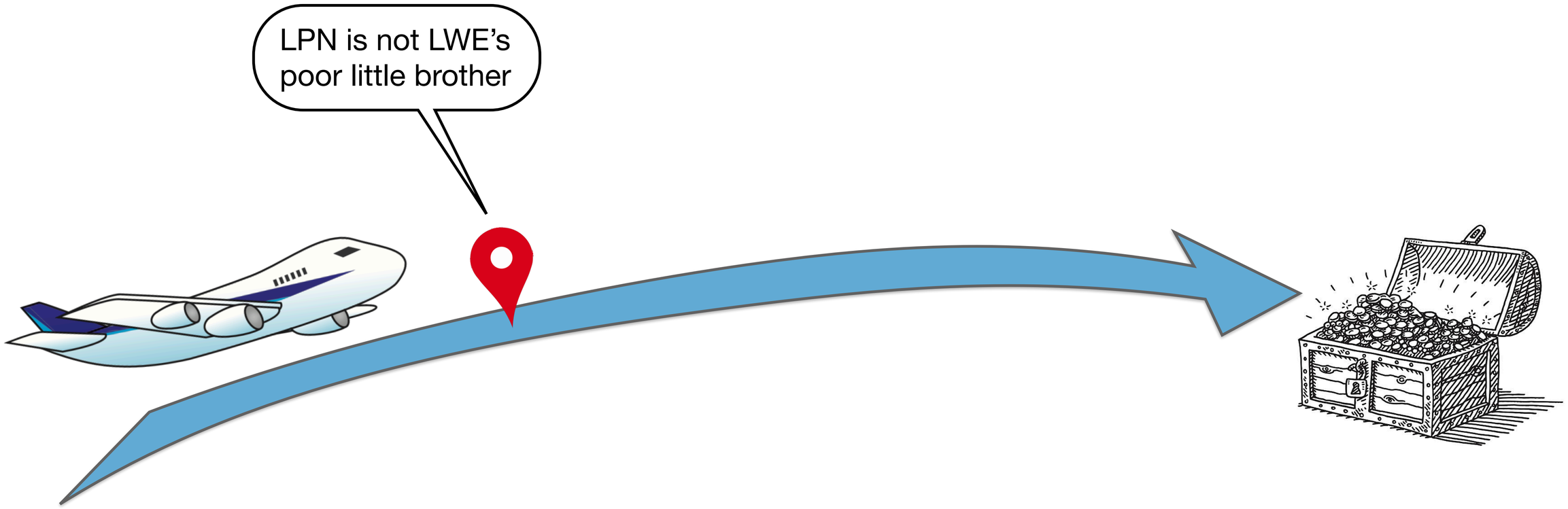

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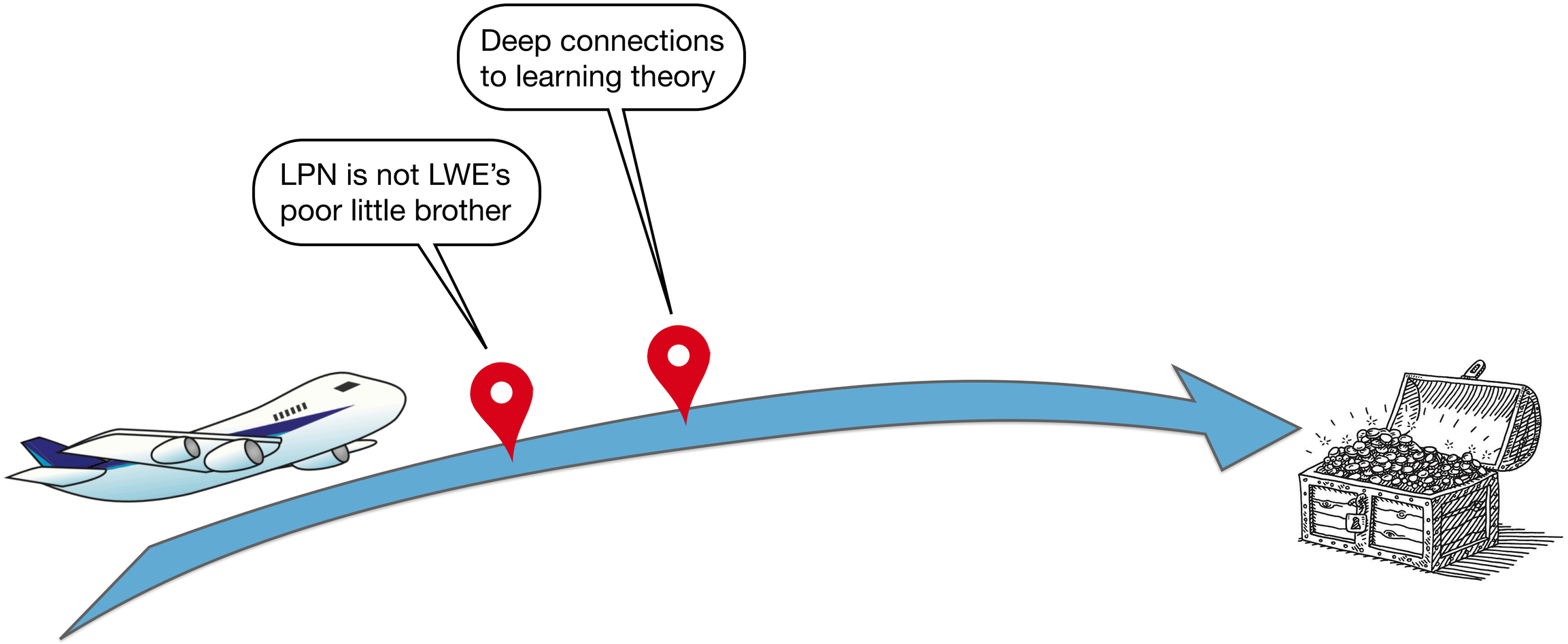

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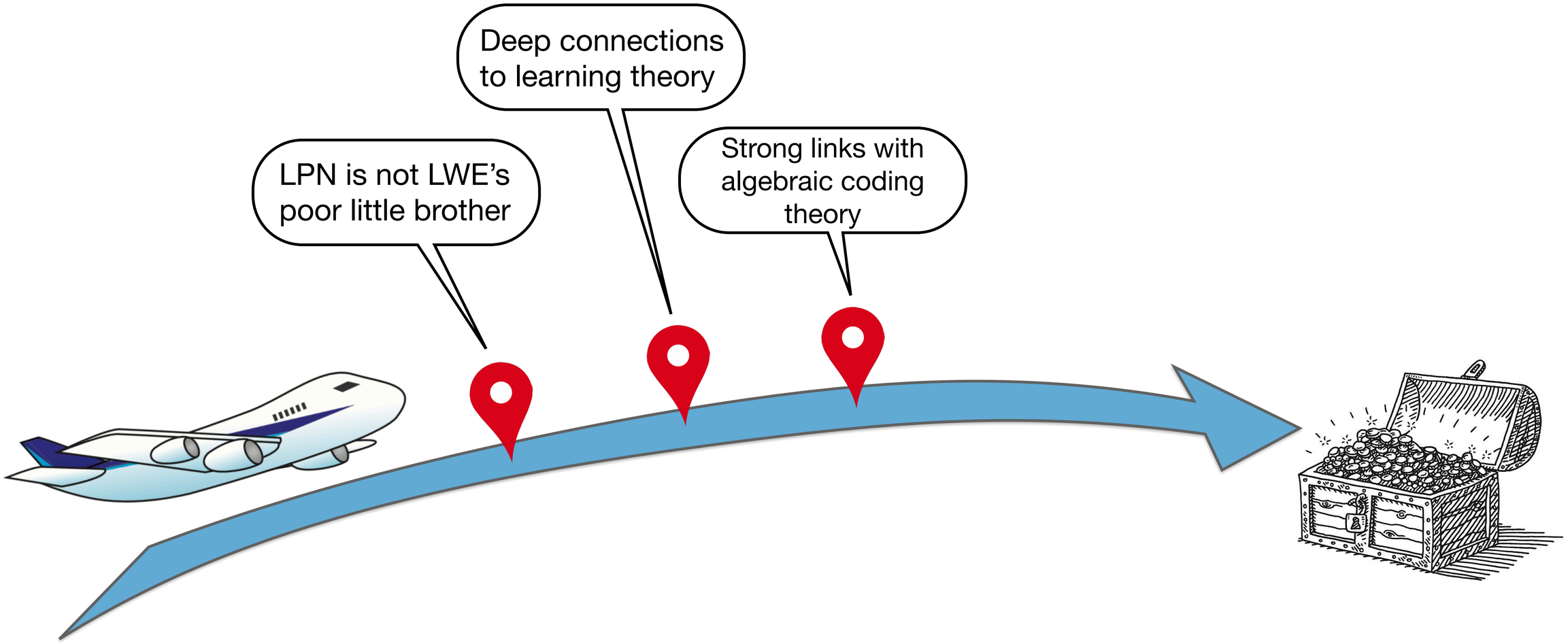

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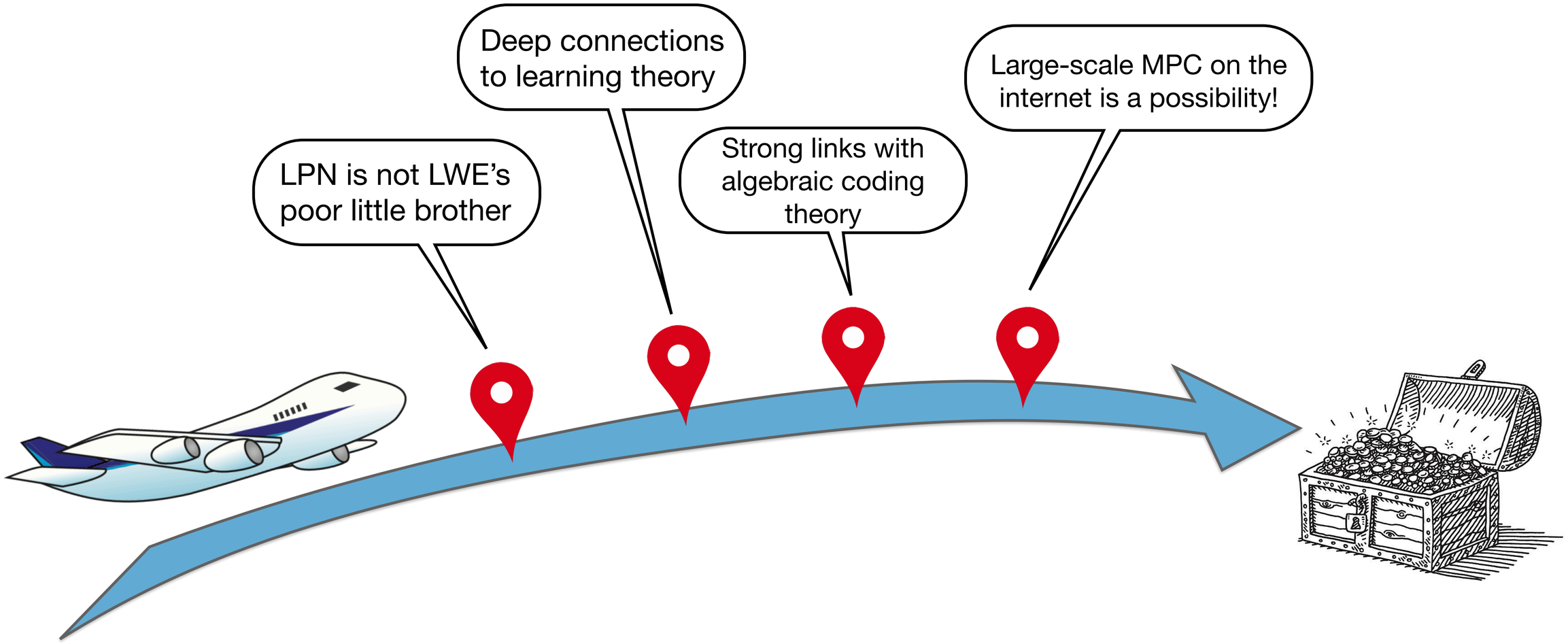

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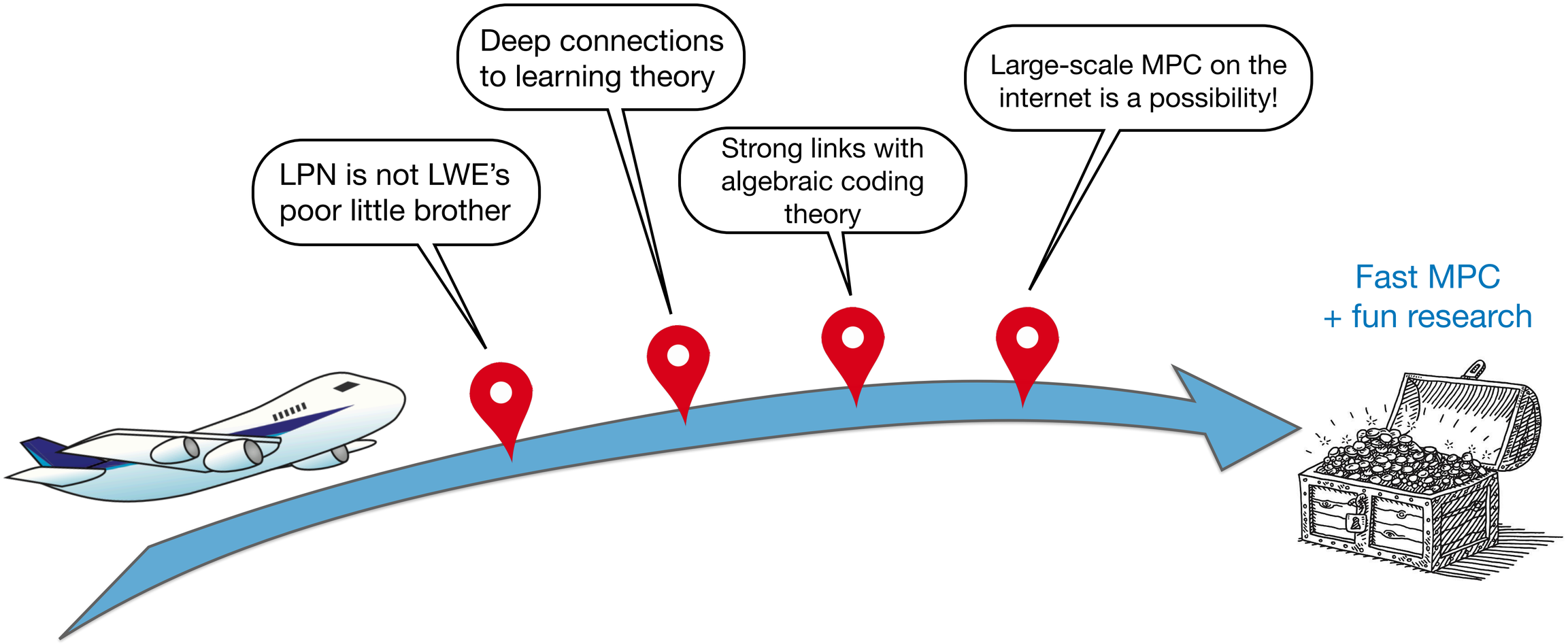
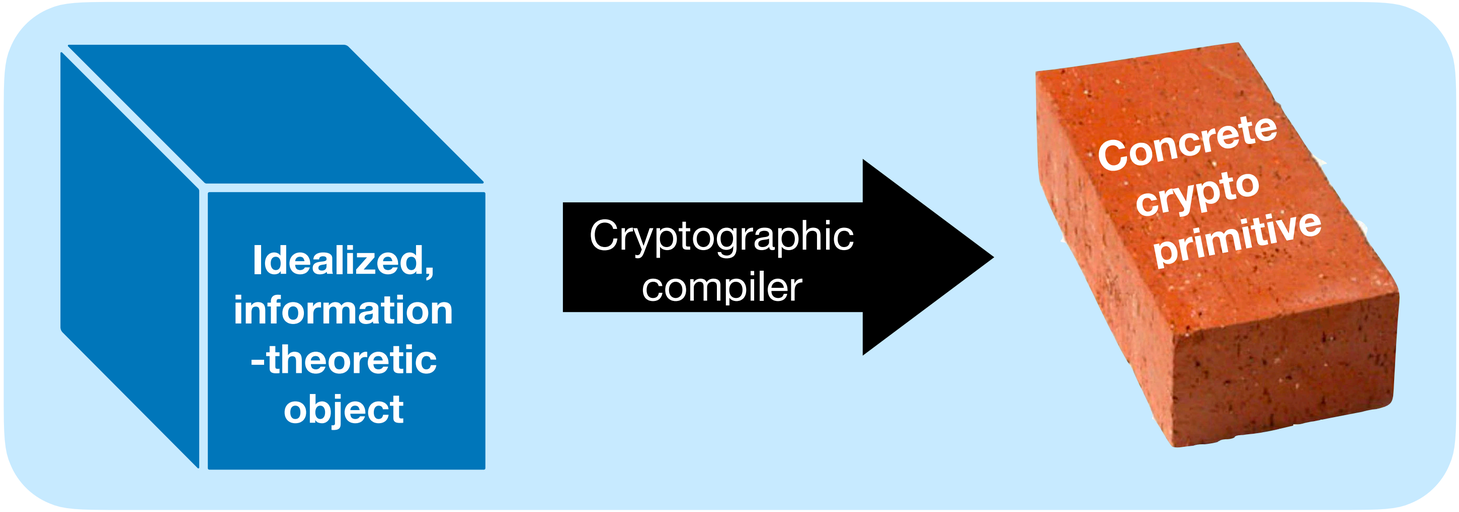


A Standard Cryptographic Approach



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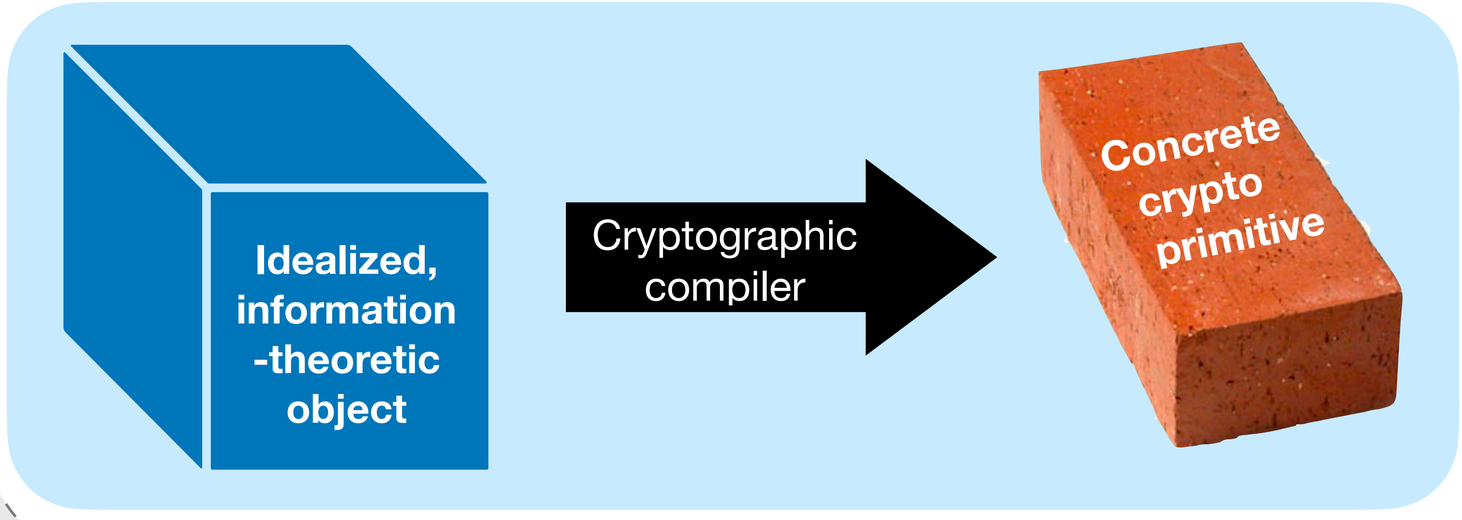

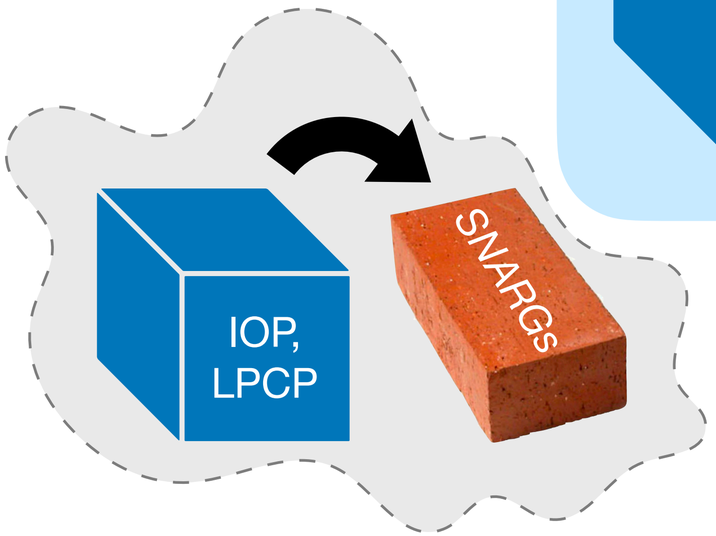

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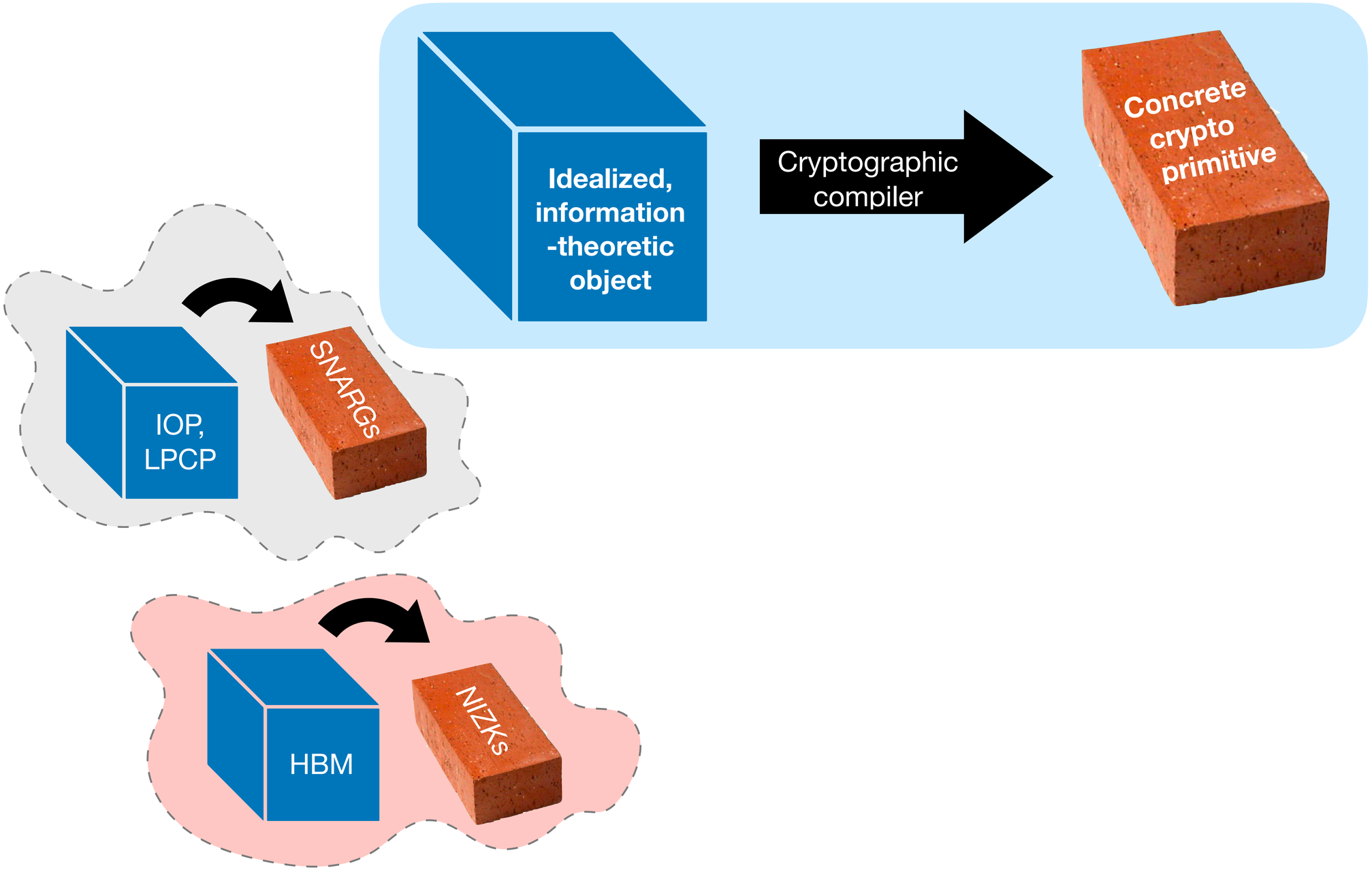

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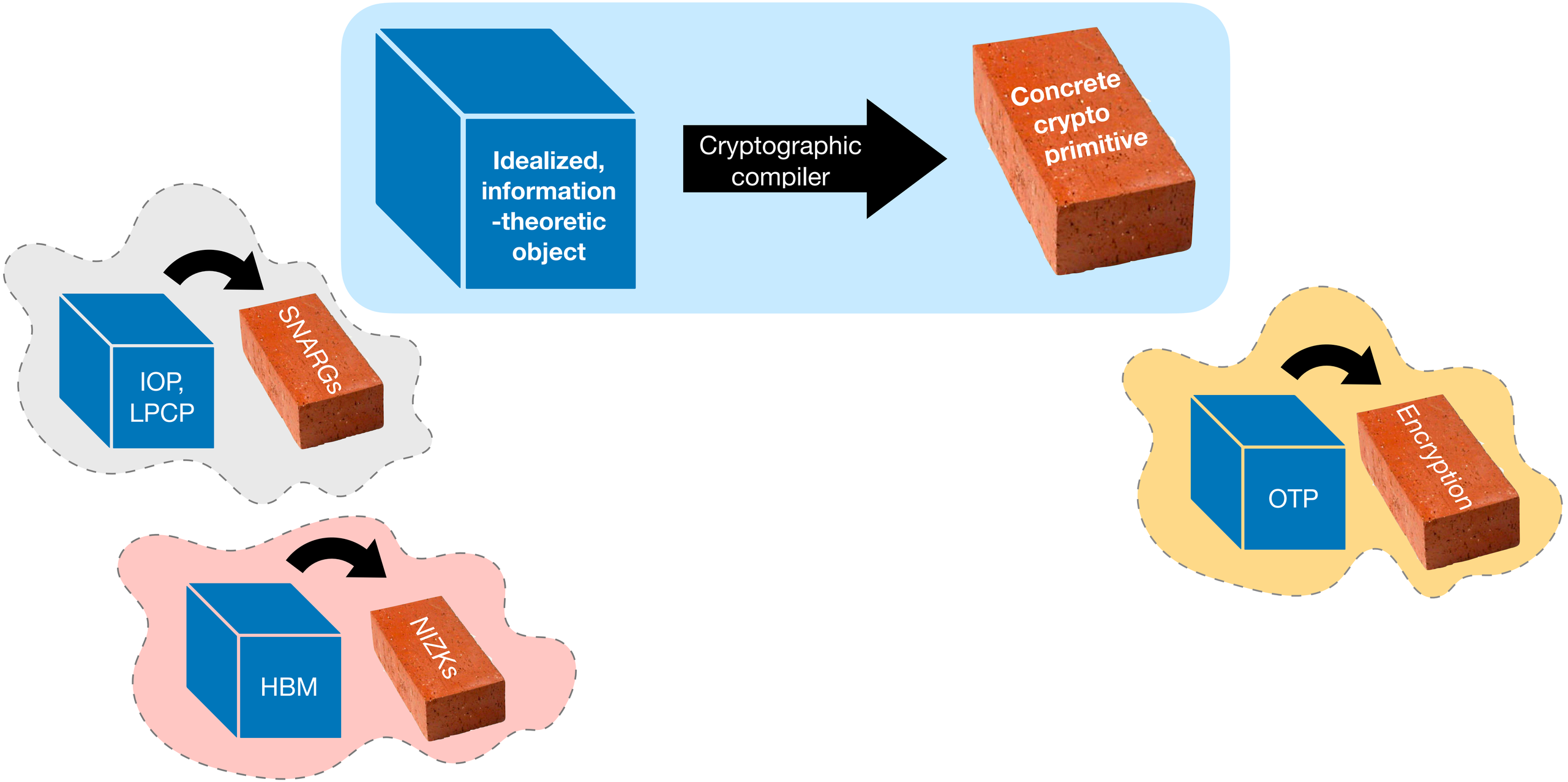

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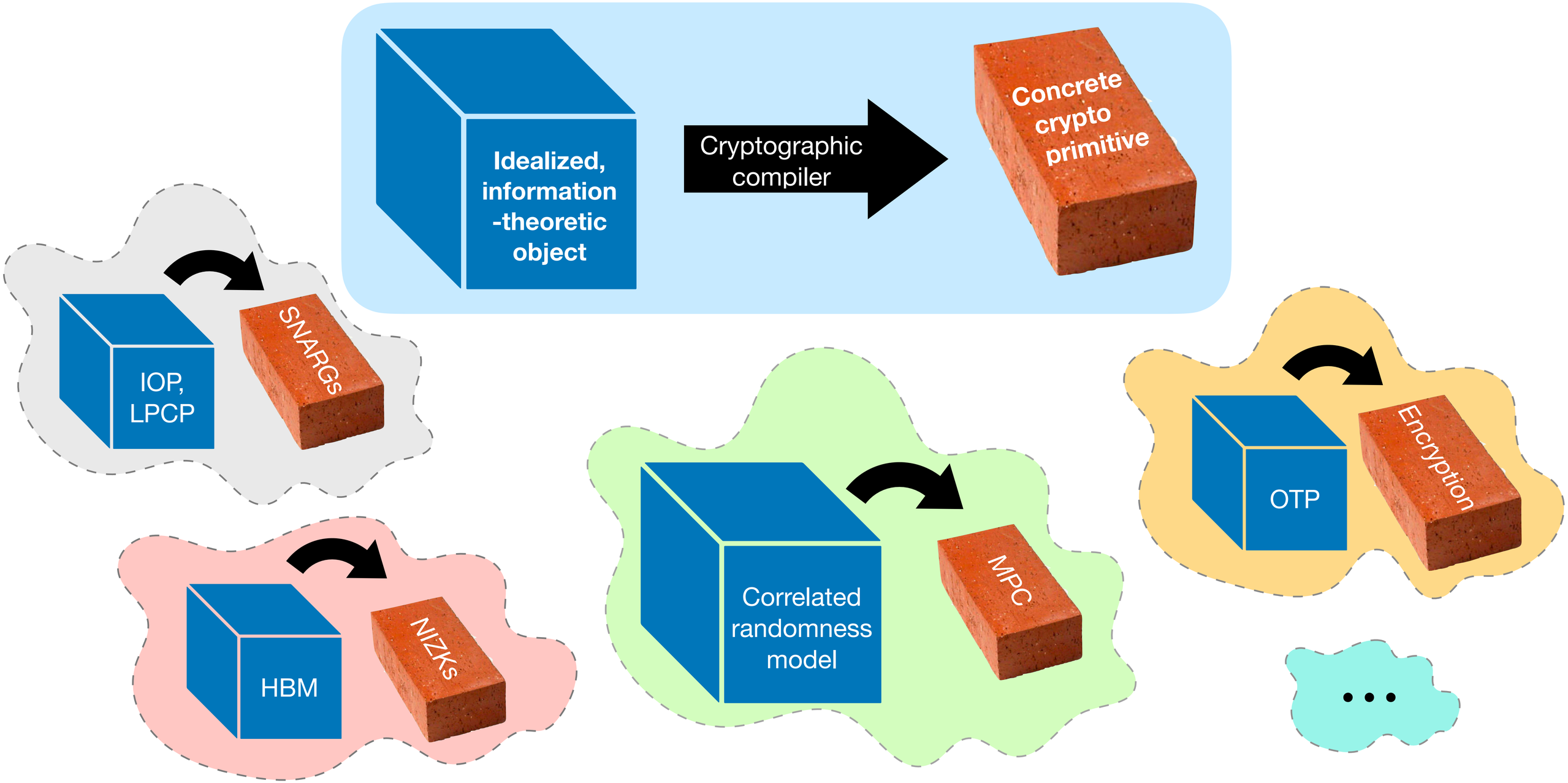

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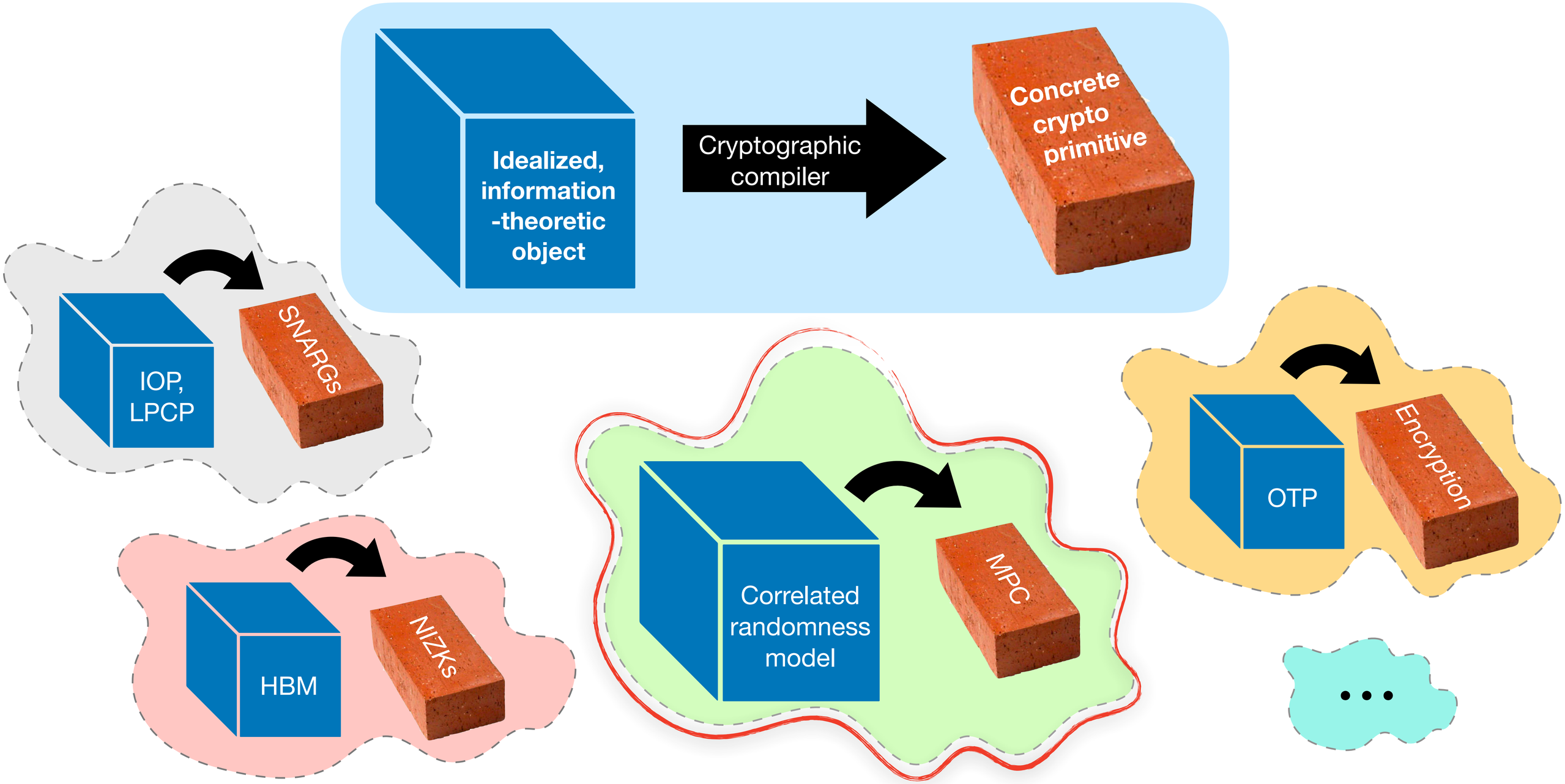
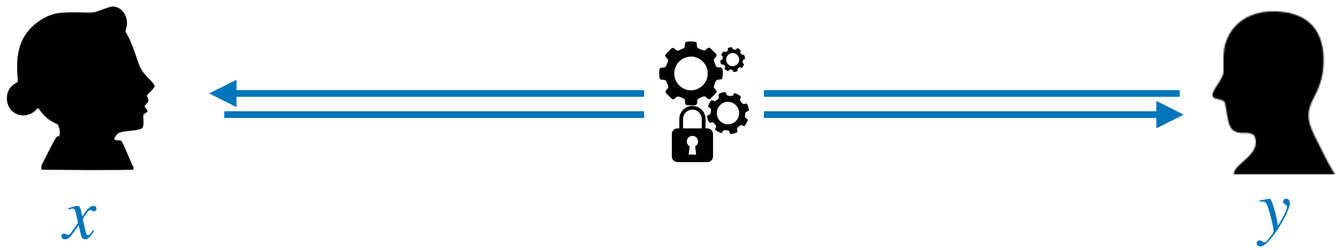


Secure Computation…

- **Goal.** C*omputing* a **public** function on secret inputs
- **Model.** n players, each with a private input x_i interacting through authenticated channels

$$f : (x, y) \mapsto (z_A, z_B)$$

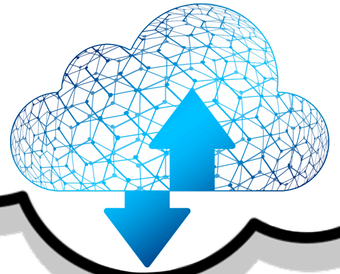


- **Output:** Alice learns z_A and Bob learn z_B
- **Security:** Alice and Bob learn nothing else

… Is a Practical Concern.

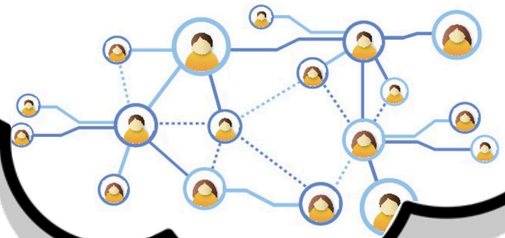
Use a dating app

Search over our Cloud storage



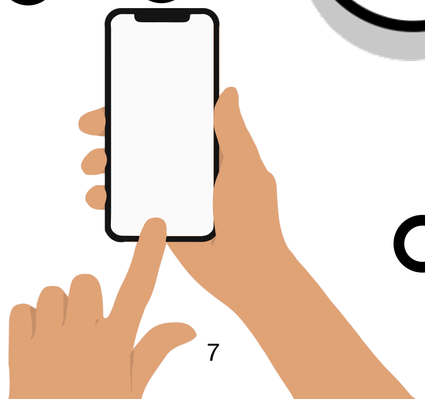
See a targeted advertising

Use a social network



Use a healthcare app

Get a recommendation on a streaming platform



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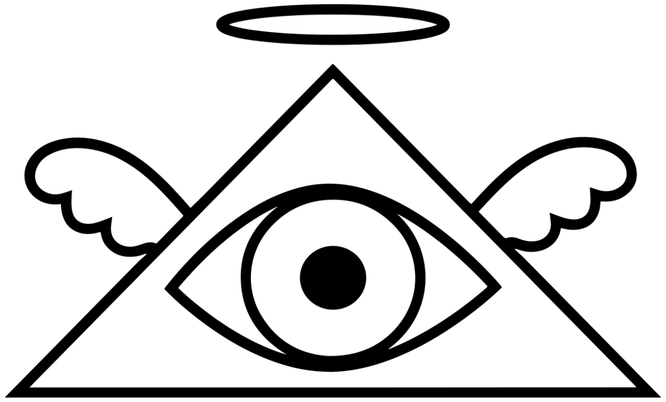
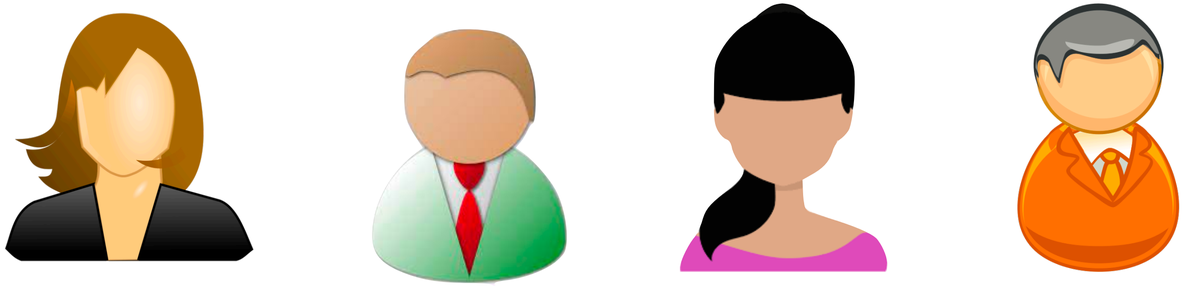
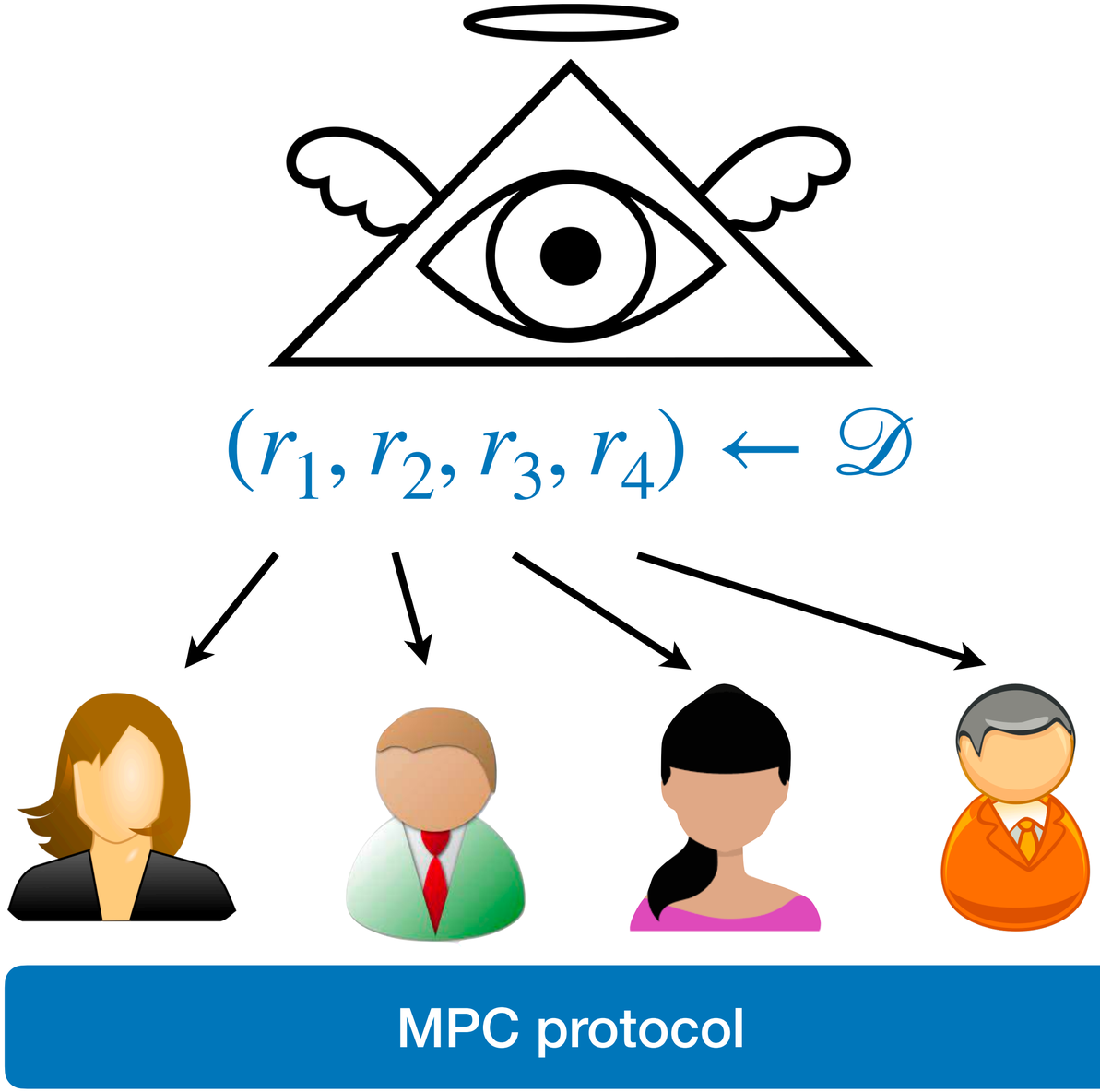
The Correlated Randomness Model





MPC protocol

The Correlated Randomness Model



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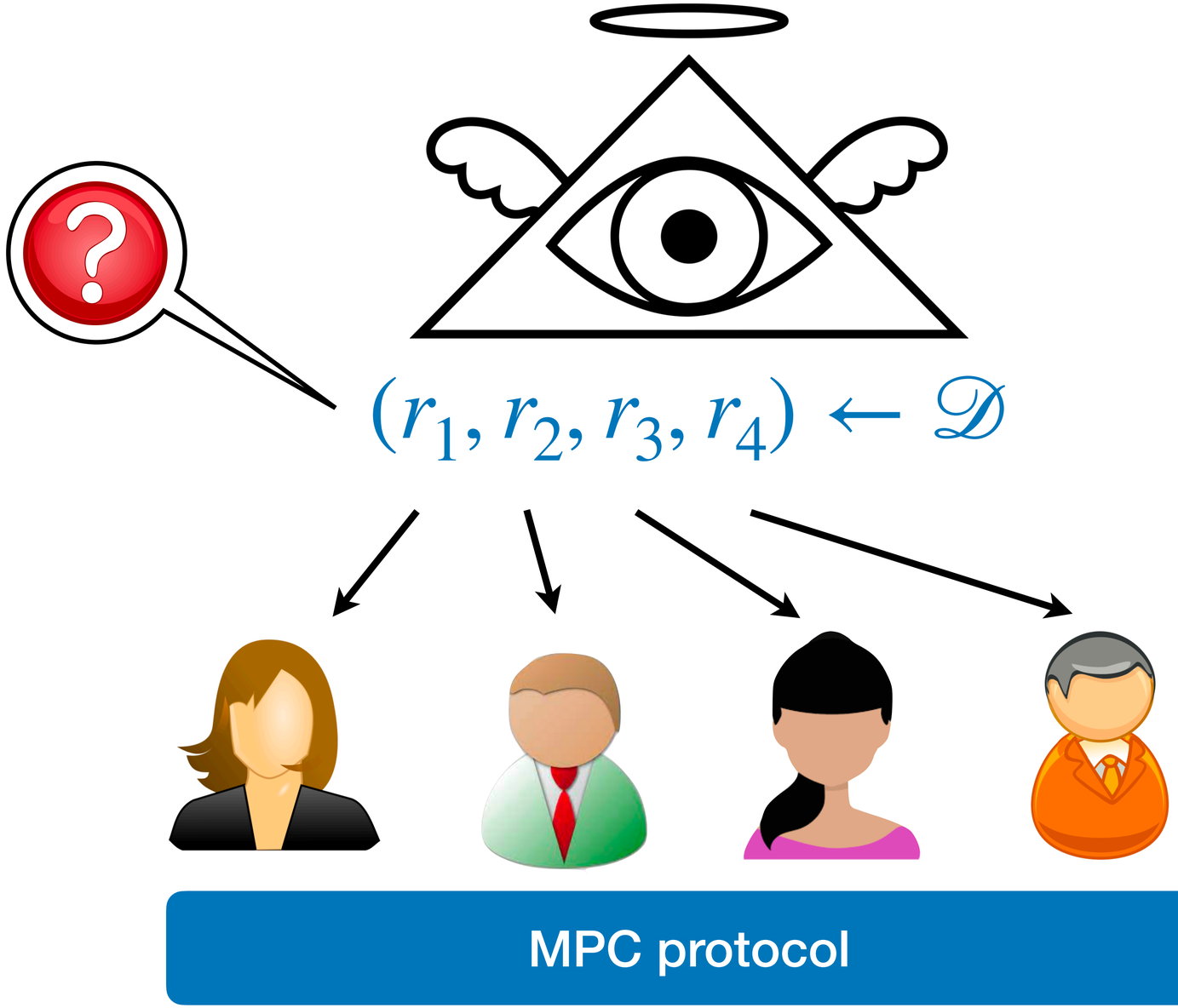

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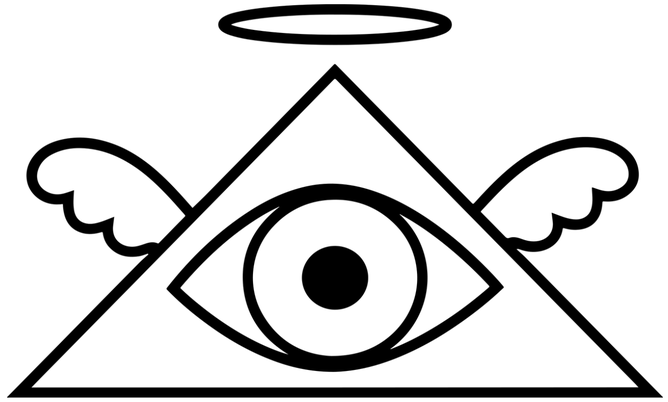
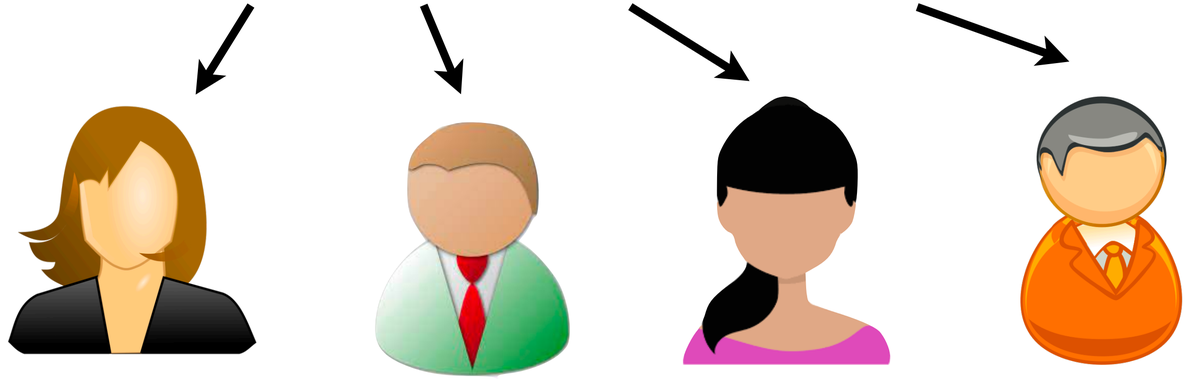


Additive correlations $\left\{ \right.$ $r \leftarrow \mathscr{D}$

$(r_1, r_2, r_3, r_4) \leftarrow \text{shares}(r)$



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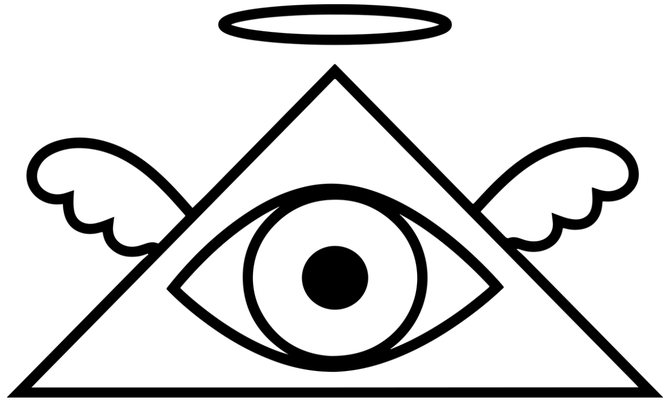
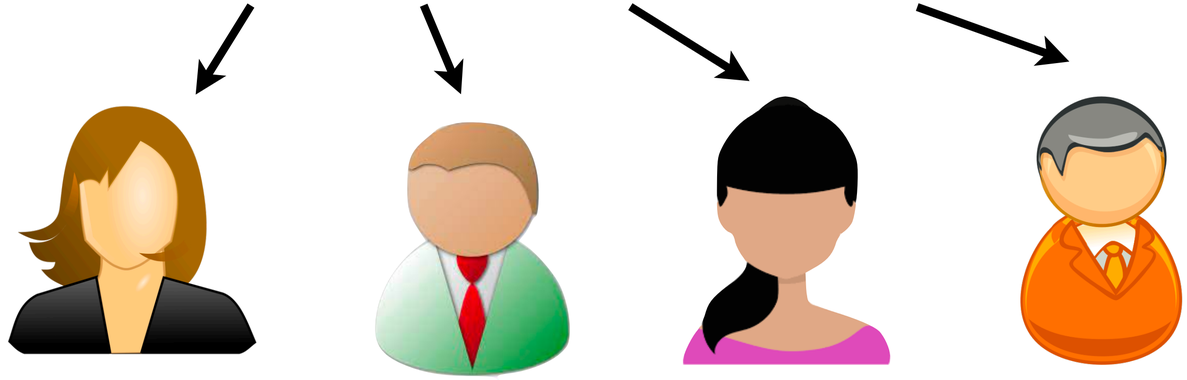


Example: Beaver triples

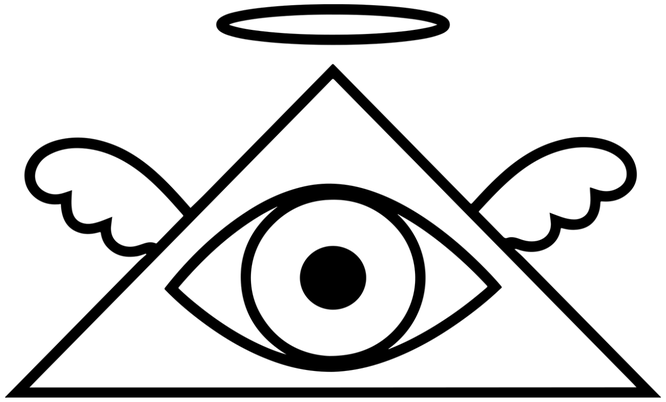
$$\begin{cases} (a_i, b_i)_{i \le n} \leftarrow (\mathbb{F}_2 \times \mathbb{F}_2)^n \\ \text{shares}((a_i, b_i, a_i \cdot b_i)_{i \le n}) \end{cases}$$



$2N$ bits / $\wedge$ gate
for N parties

GMW protocol

The Correlated Randomness Model



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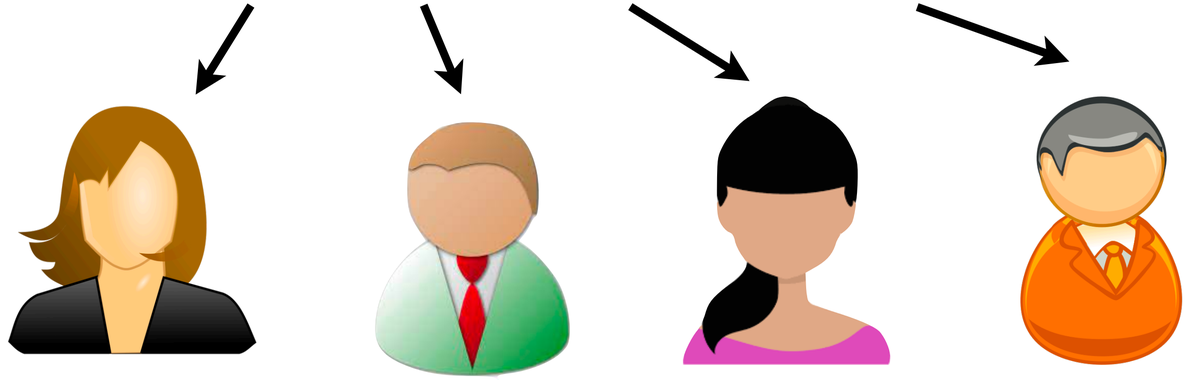

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Very practical*

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Given a correlation C, the dealer distributes shares of $C(r)$

Distributing $\langle C(r) \rangle$ *succinctly*

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Pseudorandom correlation generator

$\text{Gen}(1^\lambda) \to (\text{seed}_A, \text{seed}_B)$ **such that**

(1) $(\text{Expand}(A, \text{seed}_A), \text{Expand}(B, \text{seed}_B))$ **looks like** n **samples from the target correlation, and**

(2) $\text{Expand}(A, \text{seed}_A)$ **looks 'random conditioned on satisfying the correlation with** $\text{Expand}(B, \text{seed}_B)$**' to Bob (similar property w.r.t. Alice).**

A Template to Instantiate *Efficiently* the Correlated Randomness Model: MPC with silent preprocessing

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Preprocessing phase

Online phase

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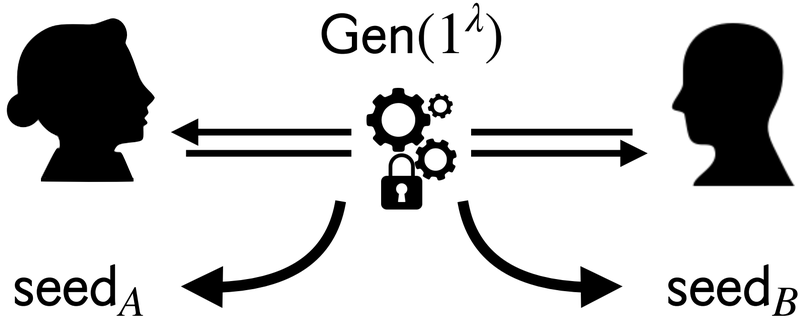
One-time short interaction



Interactive protocol with short communication and computation; Alice and Bob store a small seed afterwards.

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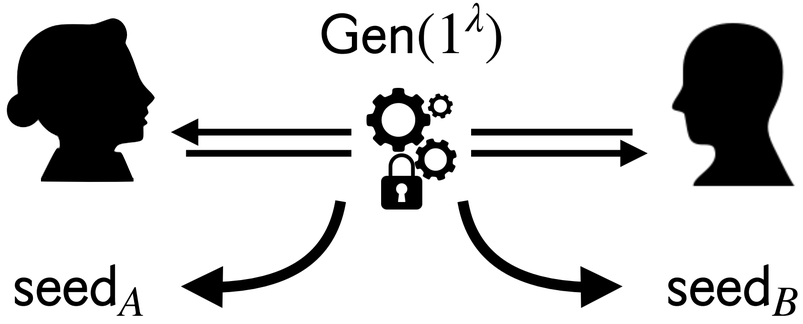

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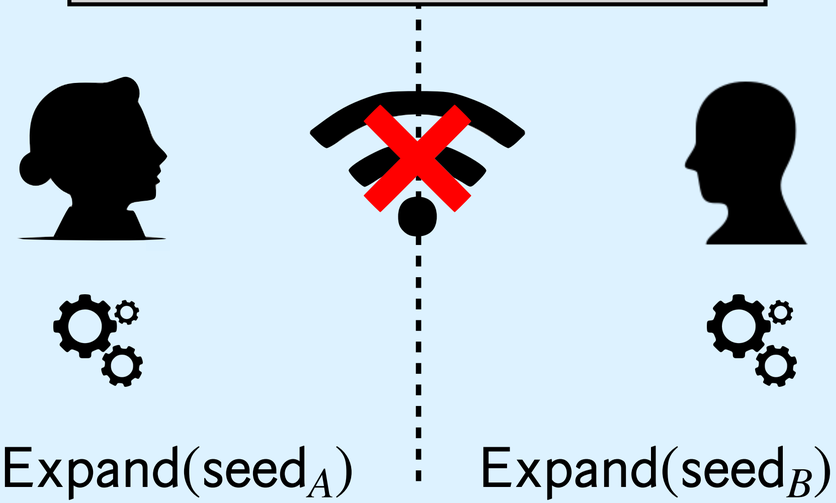
'Silent' computation



The bulk of the preprocessing phase is offline: Alice and Bob stretch their seeds into large pseudorandom correlated strings.

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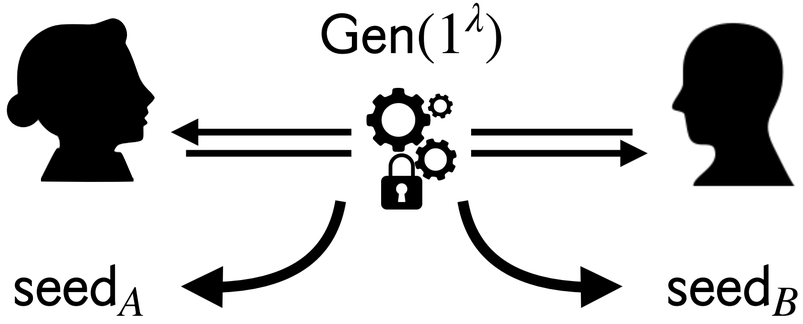

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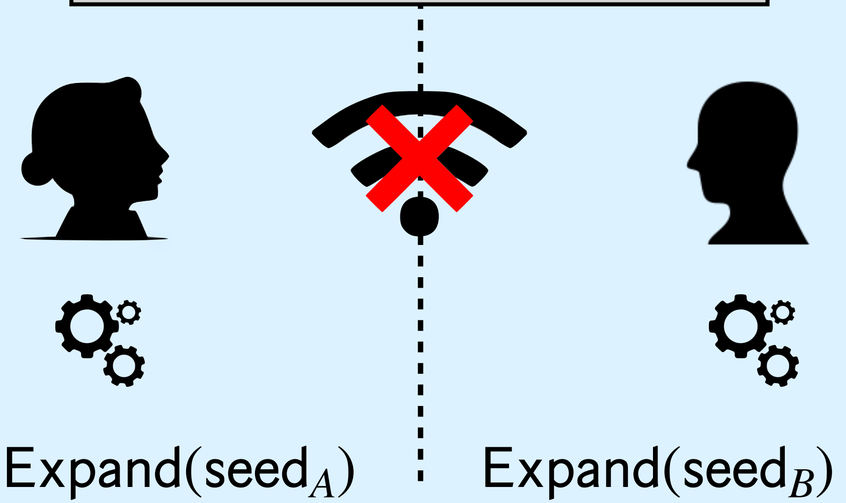

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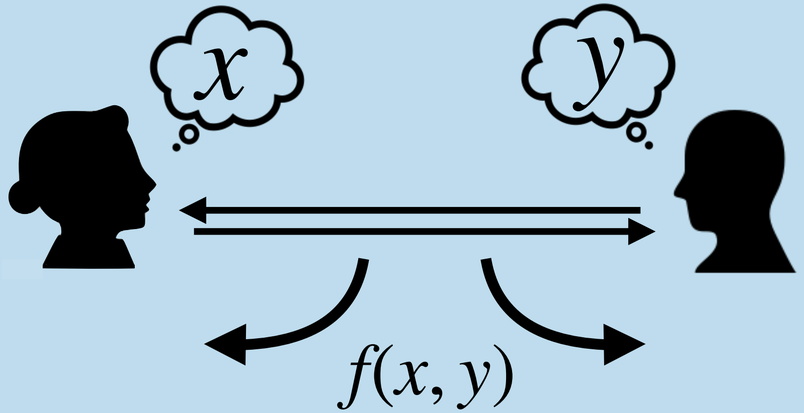
Non-cryptographic



Alice and Bob consume the preprocessing material in a fast, non-cryptographic online phase.

Preprocessing phase

Online phase

Q: What Correlations C do we Consider?

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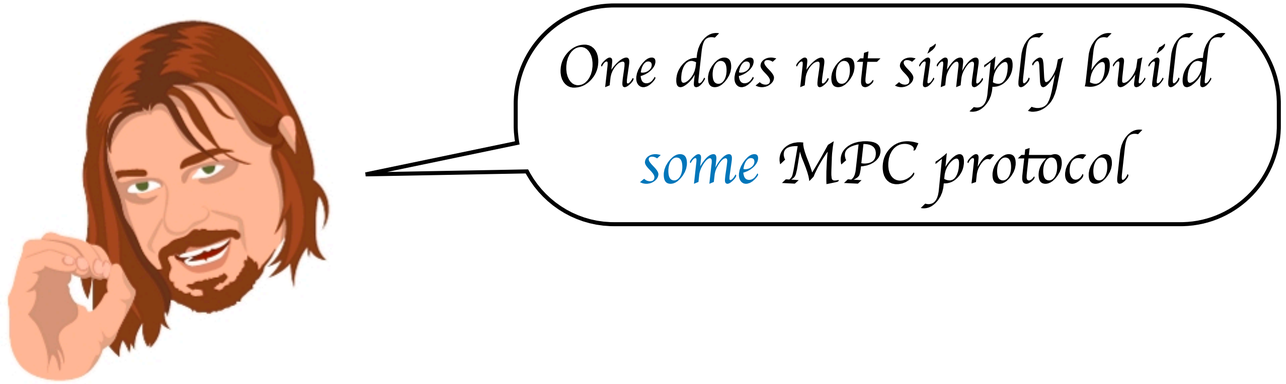

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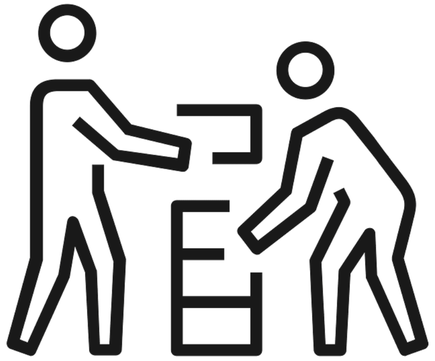
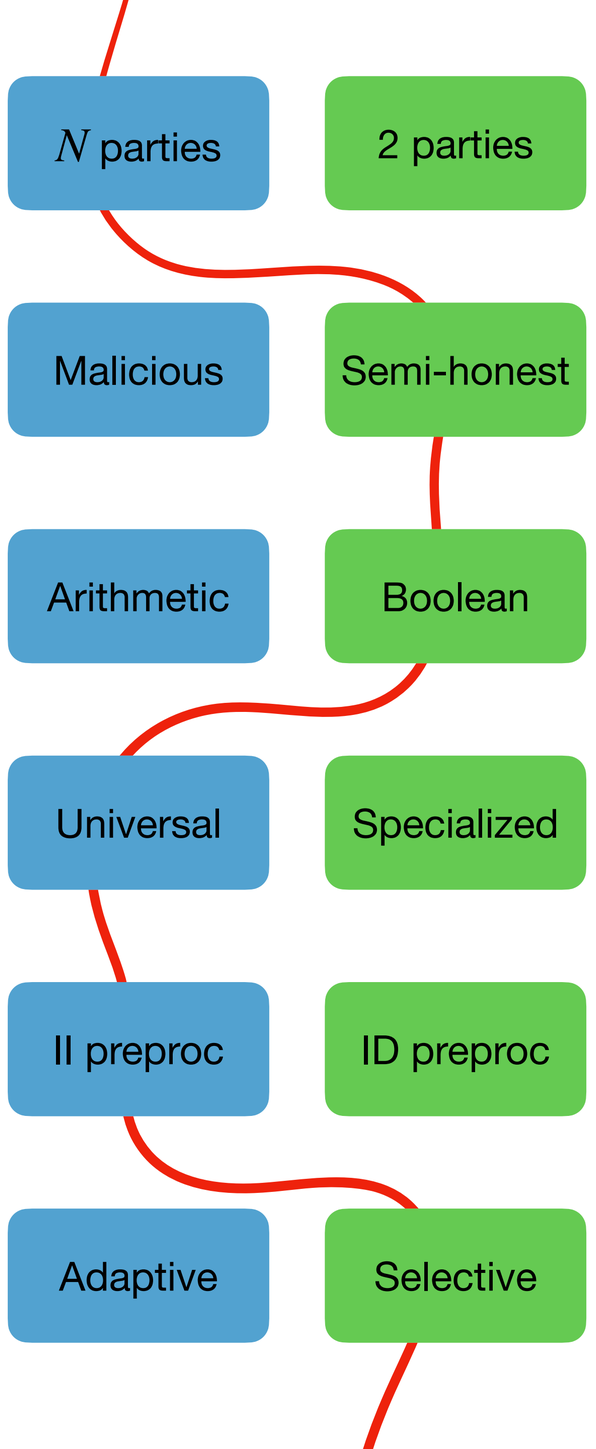
One does not simply build some MPC protocol





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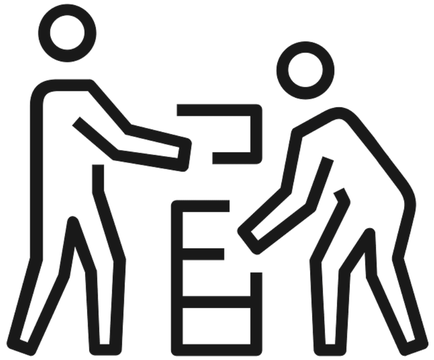
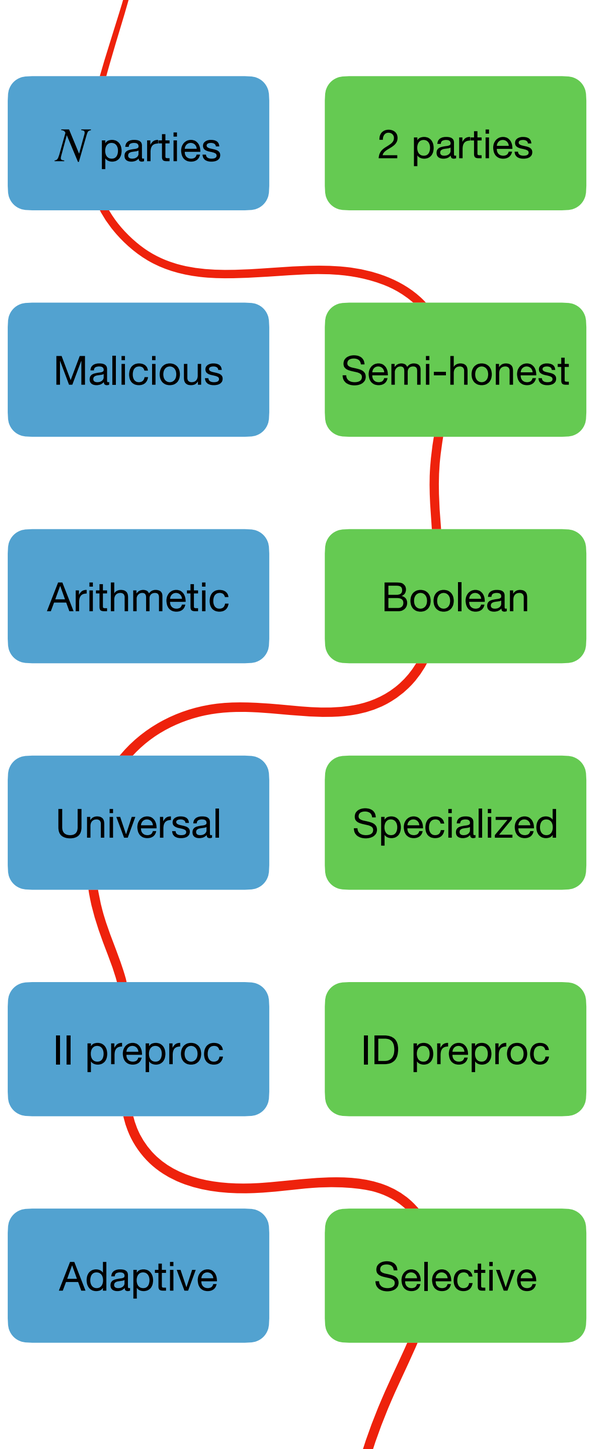
Depending on the application, you'll want:

VOLE, OT, OLE, bilinear correlations, Beaver triples, authenticated Beaver triples, daBits, circuit-dependent correlations, polynomial correlations, matrix triples, OTTT…





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Correlation (function)

$\langle C(r) \rangle$

Shares

Random coin

Public function

$\mathrm{FSS}(C \circ \mathrm{PRG}(\mathrm{seed}))$

Function secret sharing

Short seed

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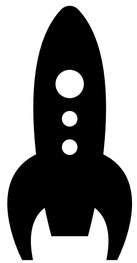
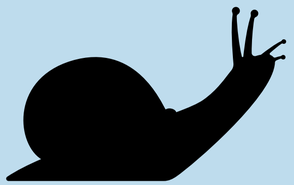
FSS $\begin{cases} \text{Share}(f) \mapsto (\ \ ,\ \ ) \\ \text{Eval}(\ \ , x) + \text{Eval}(\ \ , x) = f(x) \end{cases}$

Function Secret Sharing

FSS $\begin{cases} \text{Share}(f) \mapsto (\ \blacksquare\ \ \blacksquare\ ) \\ \text{Eval}(\ \blacksquare\ , x) + \text{Eval}(\ \blacksquare\ , x) = f(x) \end{cases}$

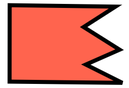
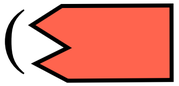
Low end	Mid end	High end
		
OWF	DDH, DCR, class groups	iO, Multi-key threshold FHE
Point functions	NC^1	All polytime functions

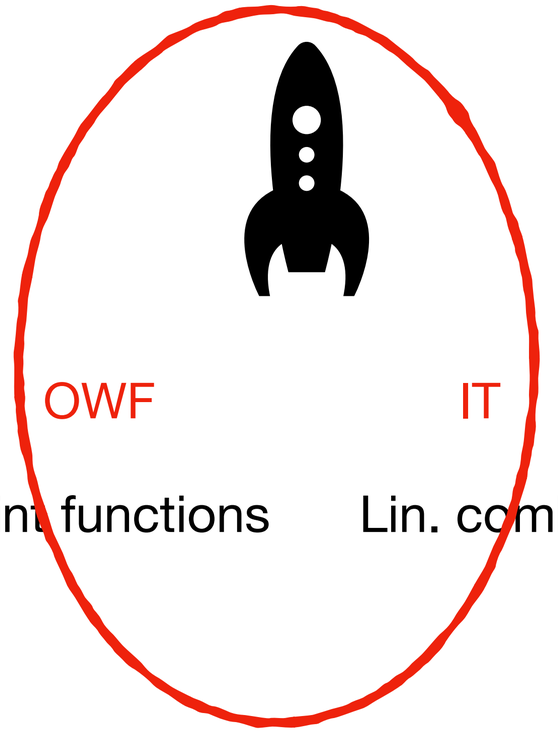
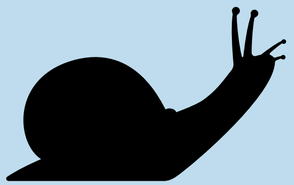
IT

Lin. comb.

Function Secret Sharing

FSS { Share(f) $\mapsto$ ()

Eval( , x) + Eval( , x) = $f(x)$

Low end	Mid end	High end
		
OWF	DDH, DCR, class groups	iO, Multi-key threshold FHE
Point functions	NC^1	All polytime functions
IT		
Lin. comb.		

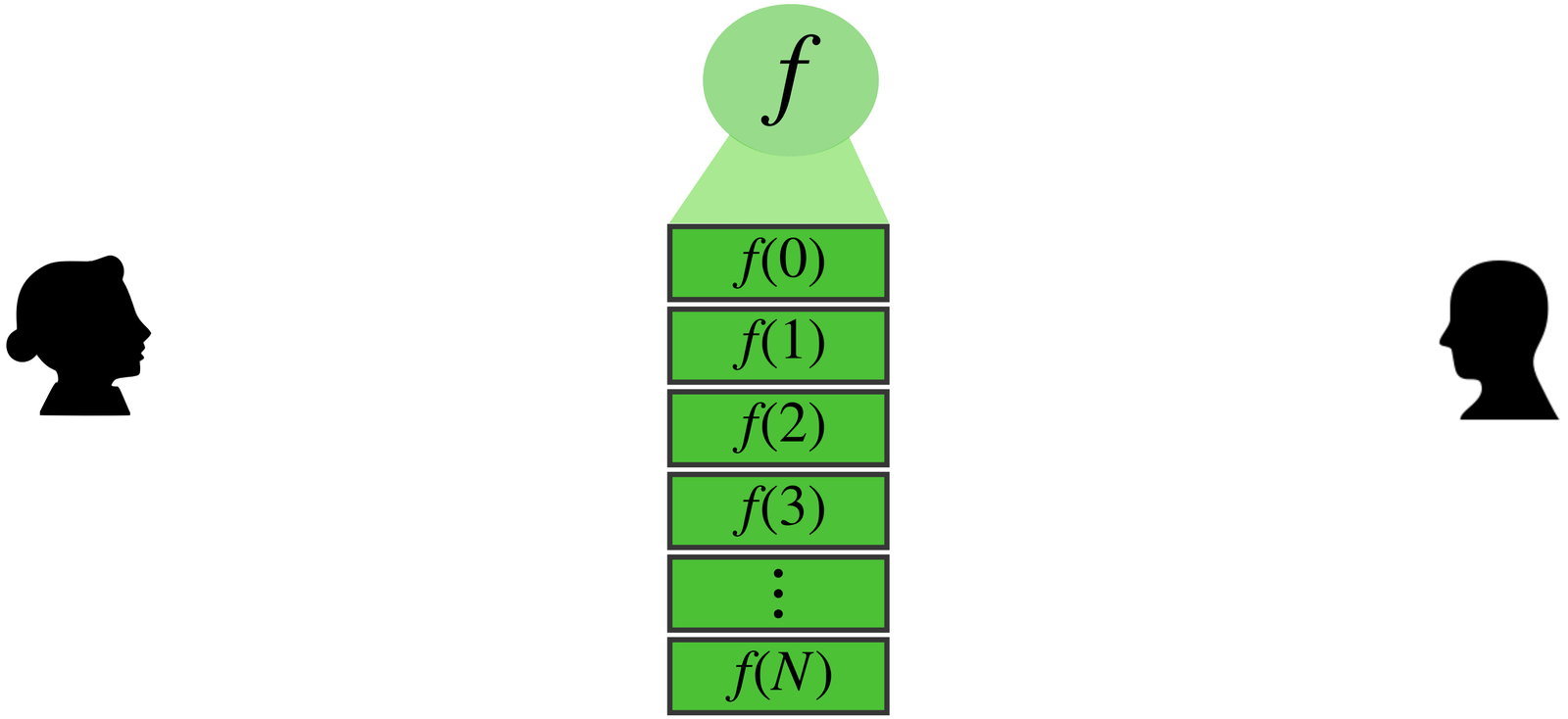
What Functions can be Shared?

Sharing an arbitrary function:

f

What Functions can be Shared?

Sharing an arbitrary function:



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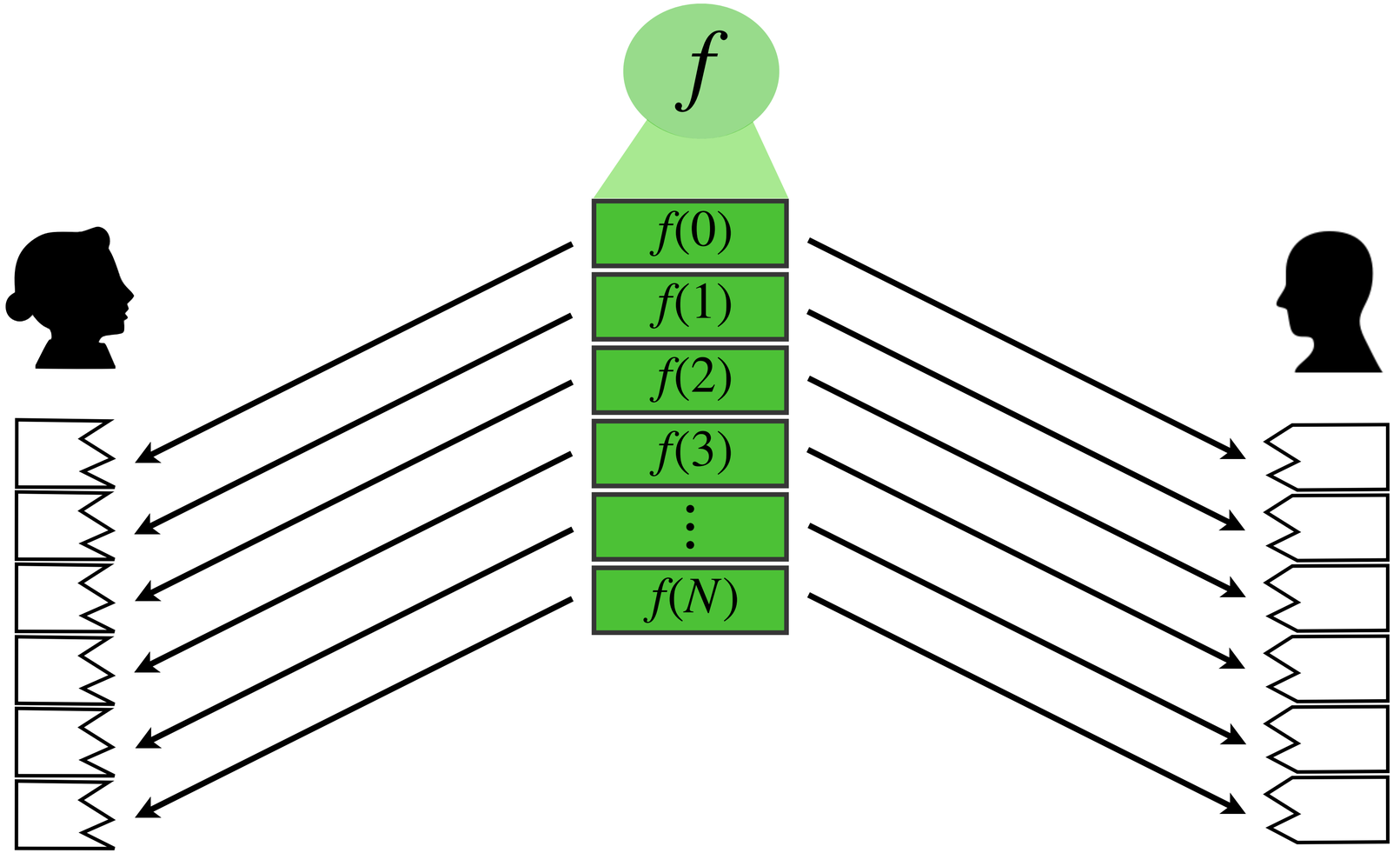

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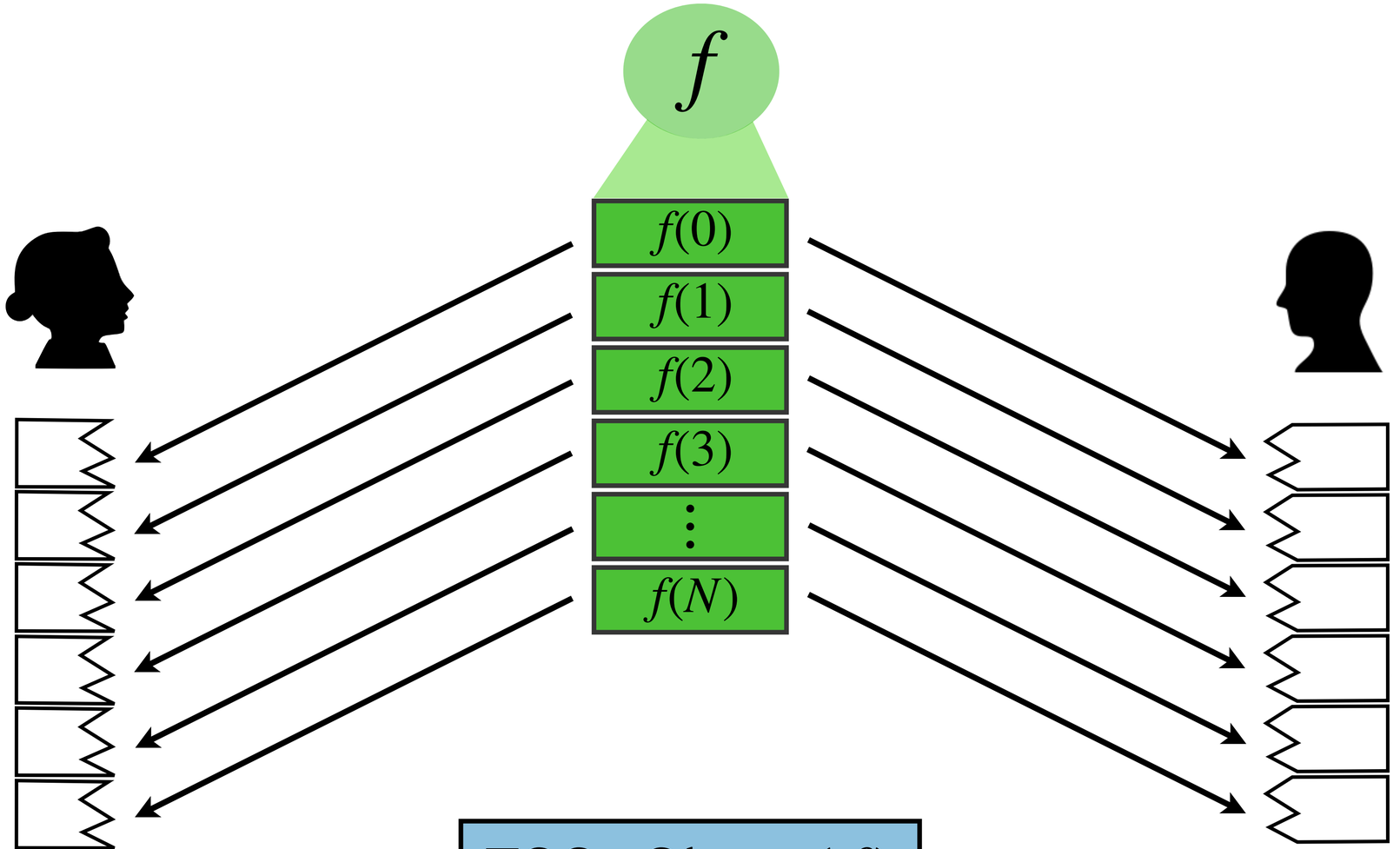

FSS . Share(f)

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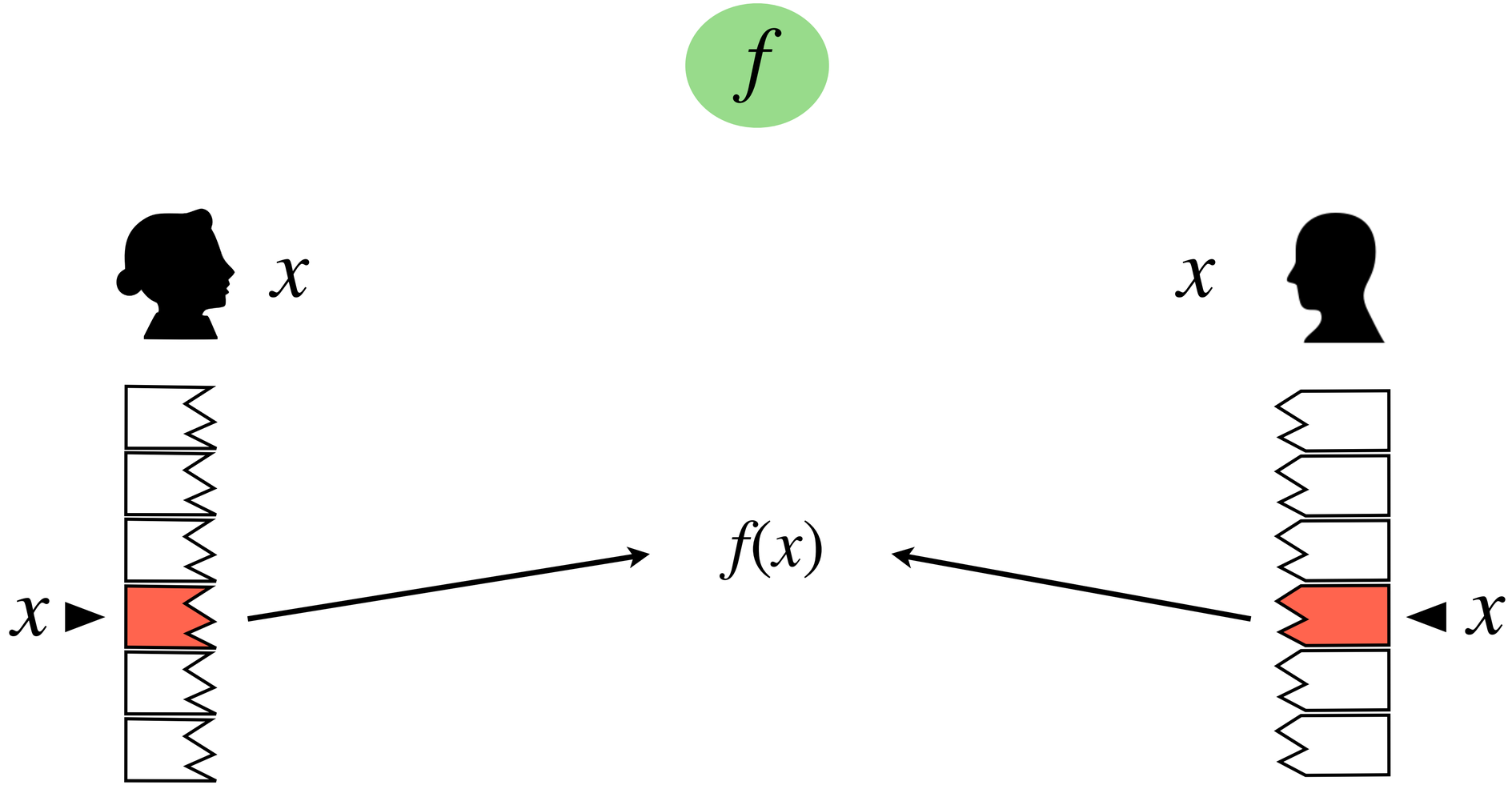

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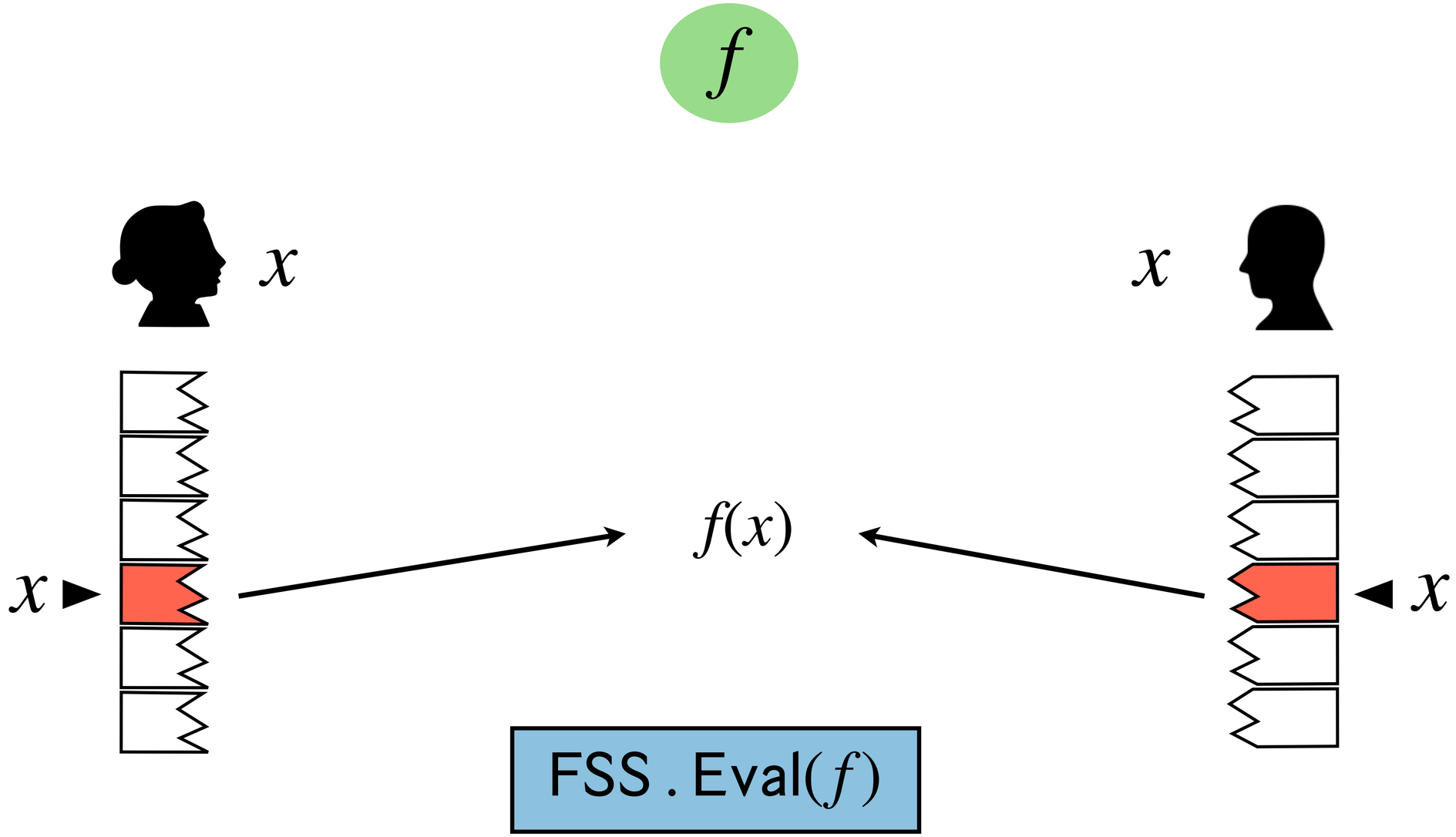

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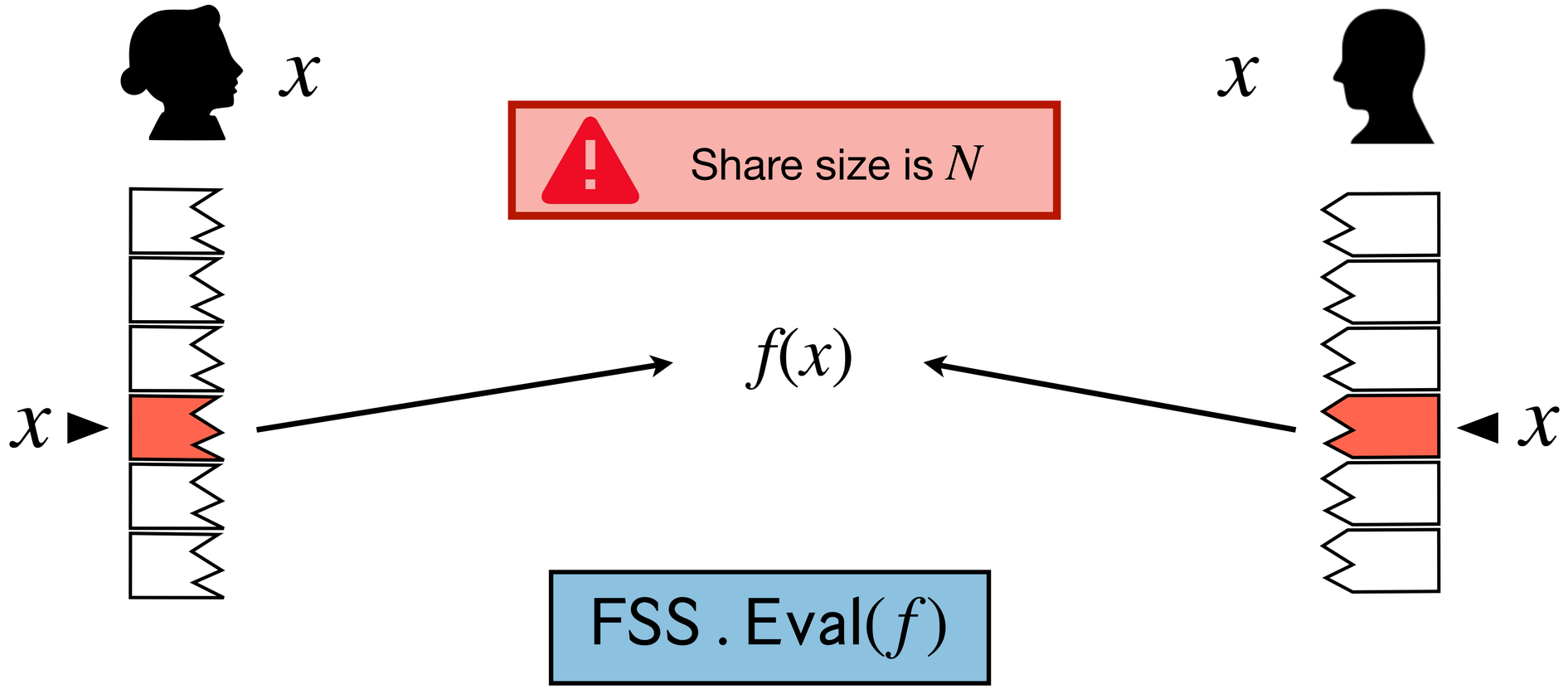

What Functions can be Shared?

succinctly

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Sharing the all zero function:

$$\forall x, f(x) = 0$$

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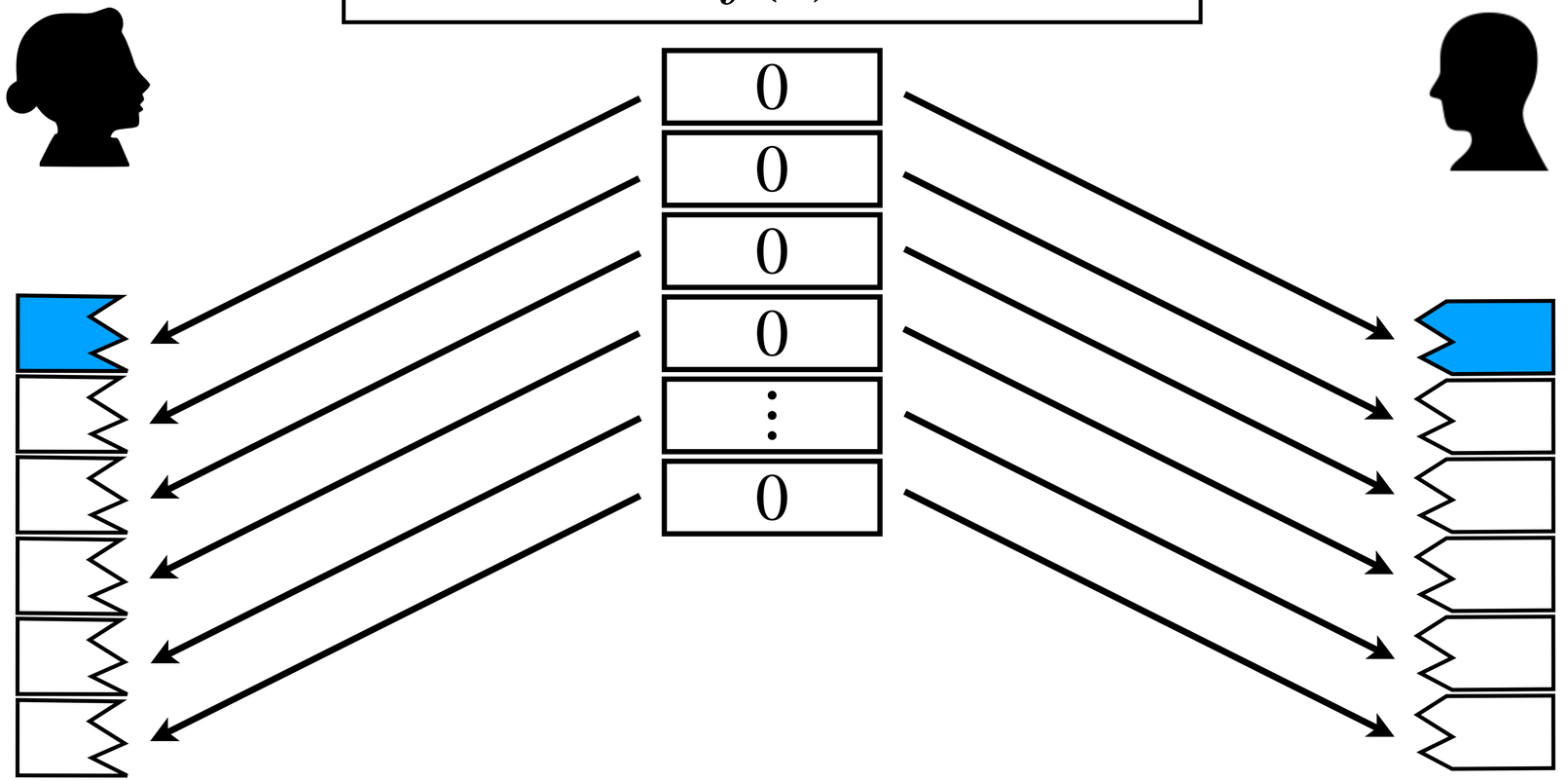

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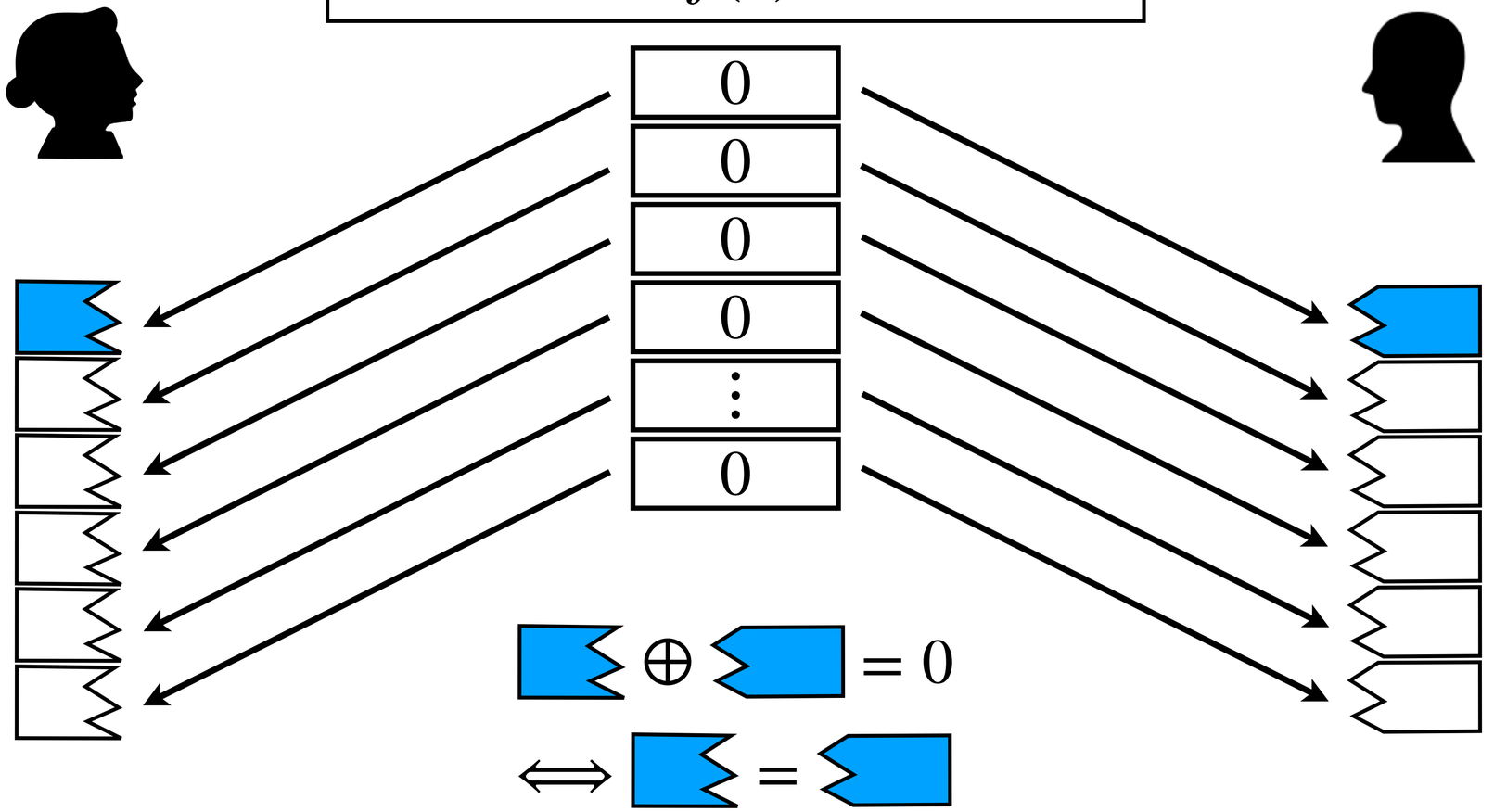

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succinctly

Sharing the all zero function:

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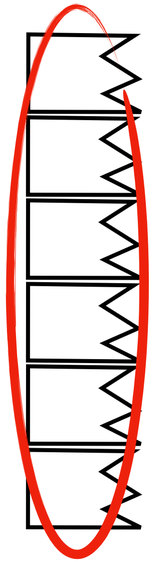
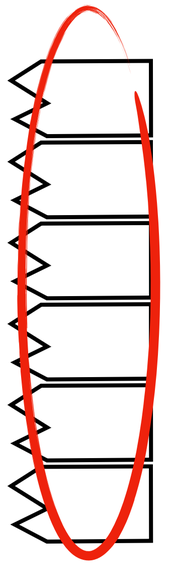
Identical long random strings





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$$f_{\alpha,\beta}(x \neq \alpha) = 0, f(\alpha) = \beta$$

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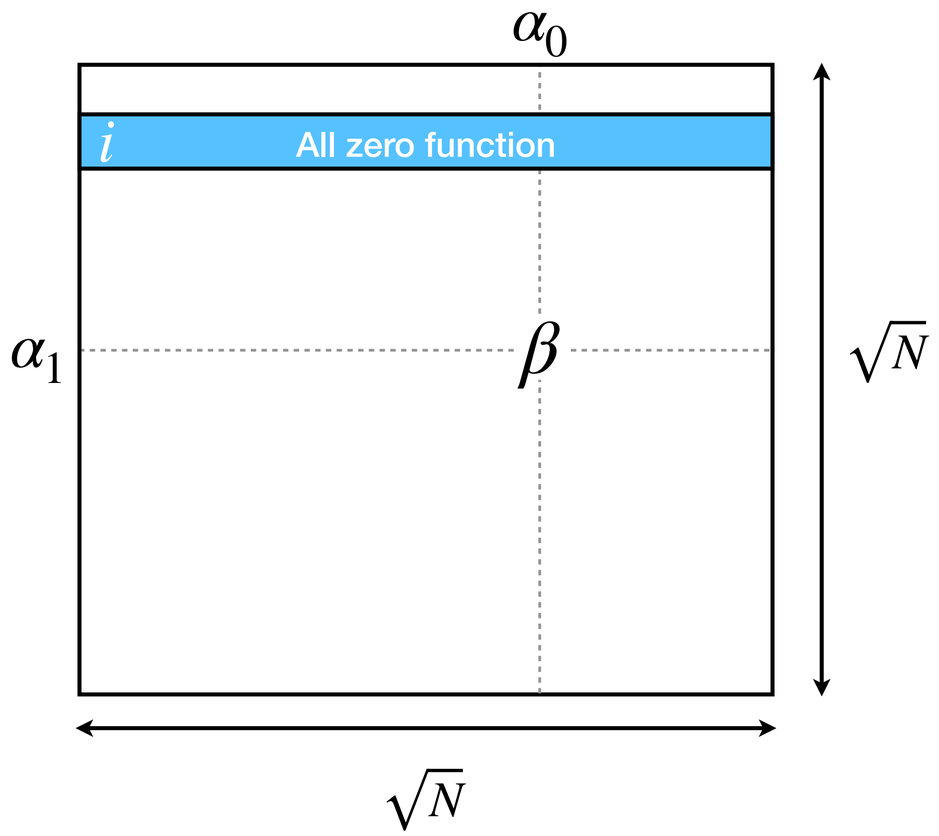

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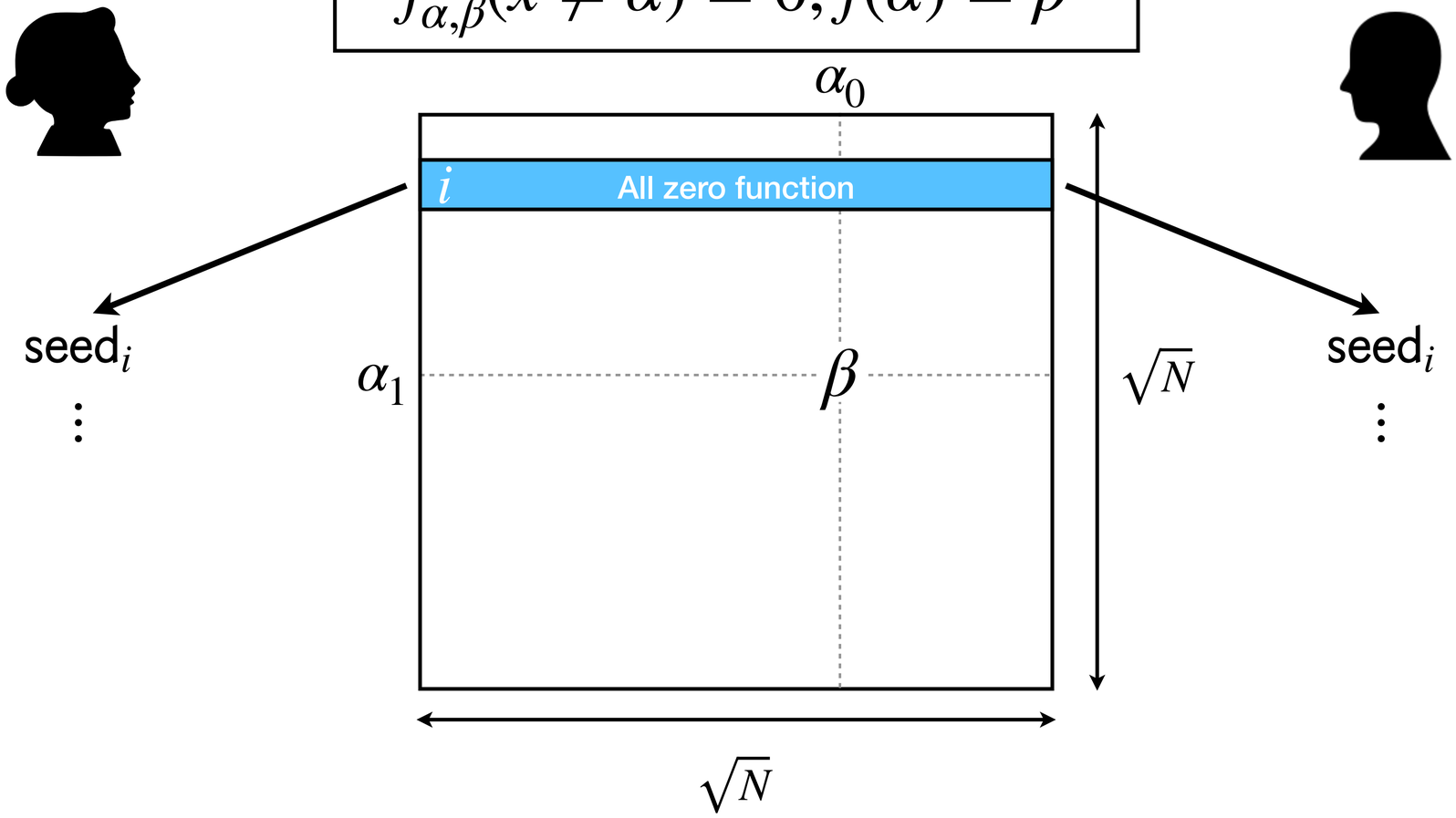

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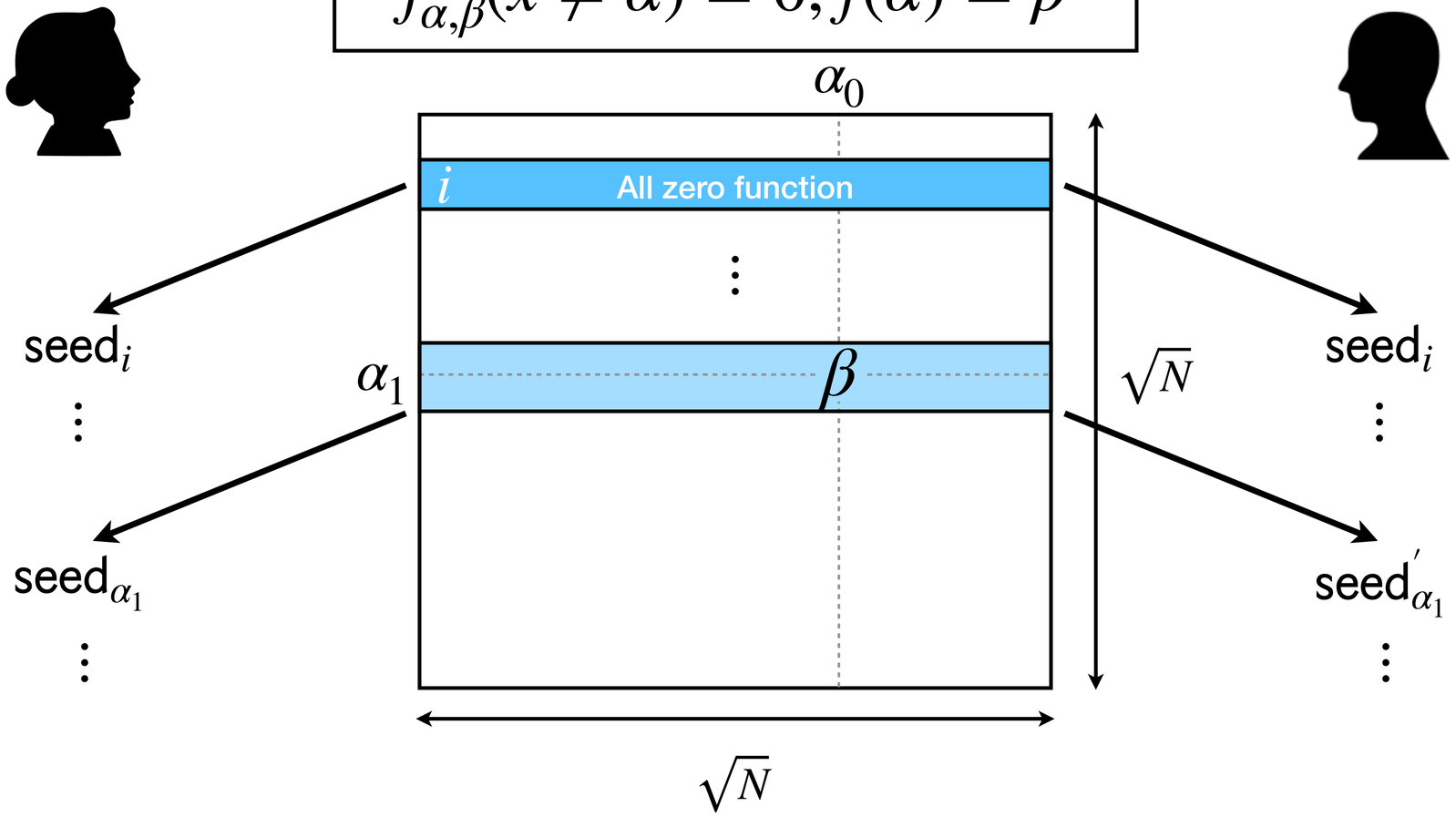

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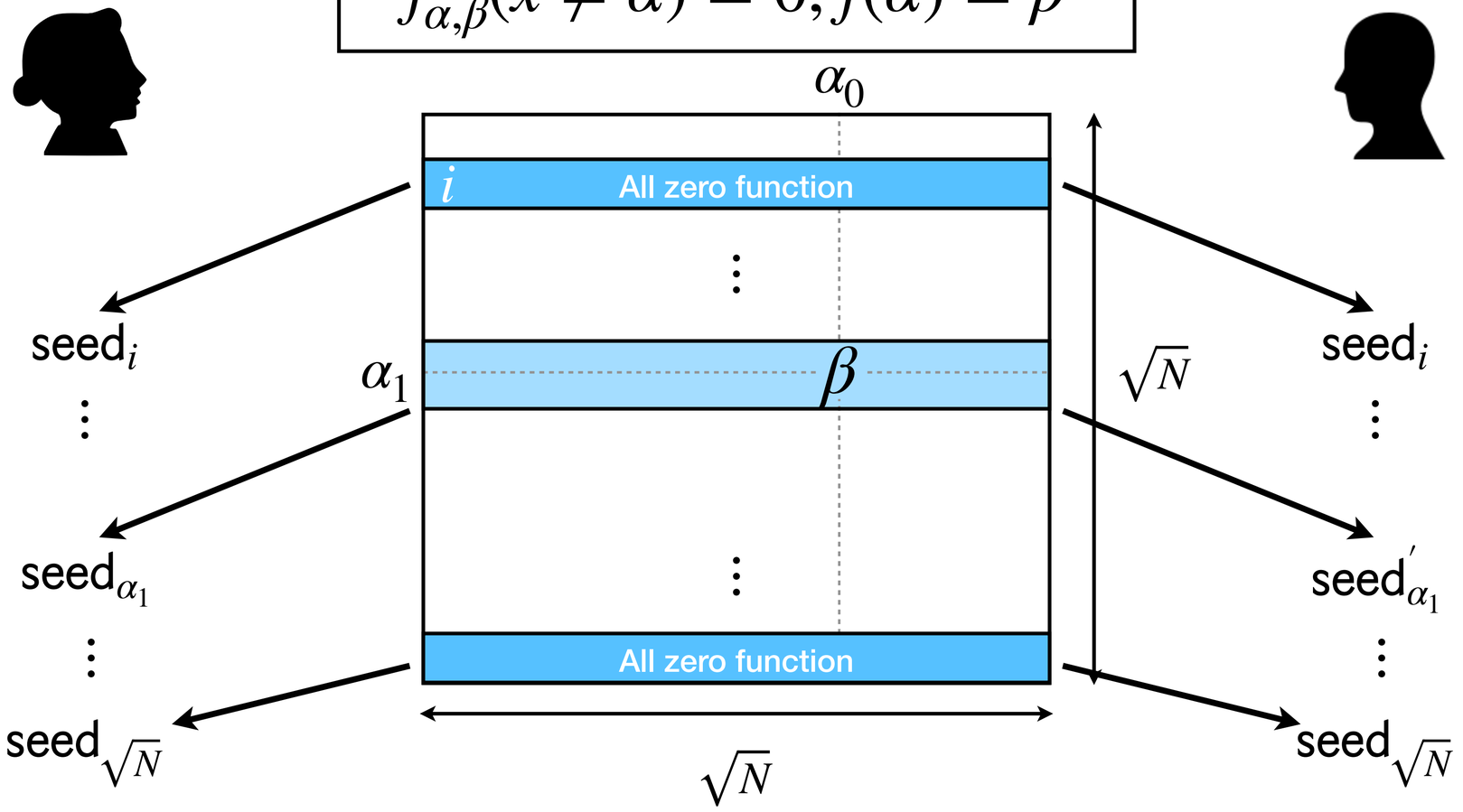

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Public

$$\Delta = \boxed{\beta} \oplus \mathrm{PRG}(\mathrm{seed}_{\alpha_1}) \oplus \mathrm{PRG}(\mathrm{seed}'_{\alpha_1})$$

seed_i

$\vdots$

seed_{α_1}

$\vdots$

$\mathrm{seed}_{\sqrt{N}}$

seed_i

$\vdots$

$\mathrm{seed}'_{\alpha_1}$

$\vdots$

$\mathrm{seed}_{\sqrt{N}}$

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b_i	seed_i	seed_i	b_i
$\vdots$		$\vdots$	
b_{α_1}	seed_{α_1}	$\mathrm{seed}'_{\alpha_1}$	$1 - b_{\alpha_1}$
$\vdots$		$\vdots$	
$b_{\sqrt{N}}$	$\mathrm{seed}_{\sqrt{N}}$	$\mathrm{seed}_{\sqrt{N}}$	$b_{\sqrt{N}}$

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$\vdots$		$\vdots$	
b_{α_1}	seed_{α_1}	$\mathrm{seed}'_{\alpha_1}$	$1 - b_{\alpha_1}$
$\vdots$		$\vdots$	
$b_{\sqrt{N}}$	$\mathrm{seed}_{\sqrt{N}}$	$\mathrm{seed}_{\sqrt{N}}$	$b_{\sqrt{N}}$

Eval :

$$\mathrm{PRG}(\mathrm{seed}_j) \oplus \Delta \cdot b_j$$

What Functions can be Shared?

succinctly

Sharing a *point function*

$$f_{\alpha,\beta}(x \neq \alpha) = 0, f(\alpha) = \beta$$

Recursing:

Giving Δ and sharing the b_i's are both essentially sharing a $\sqrt{N}$-size point function again: we can recurse the process!

b_i seed_i

$\vdots$

b_{α_1} seed_{α_1}

$\vdots$

$b_{\sqrt{N}}$ $\text{seed}_{\sqrt{N}}$

seed_i b_i

$\vdots$

seed'_{α_1} $1 - b_{\alpha_1}$

$\vdots$

$\text{seed}_{\sqrt{N}}$ $b_{\sqrt{N}}$

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b_i	seed_i	seed_i	b_i
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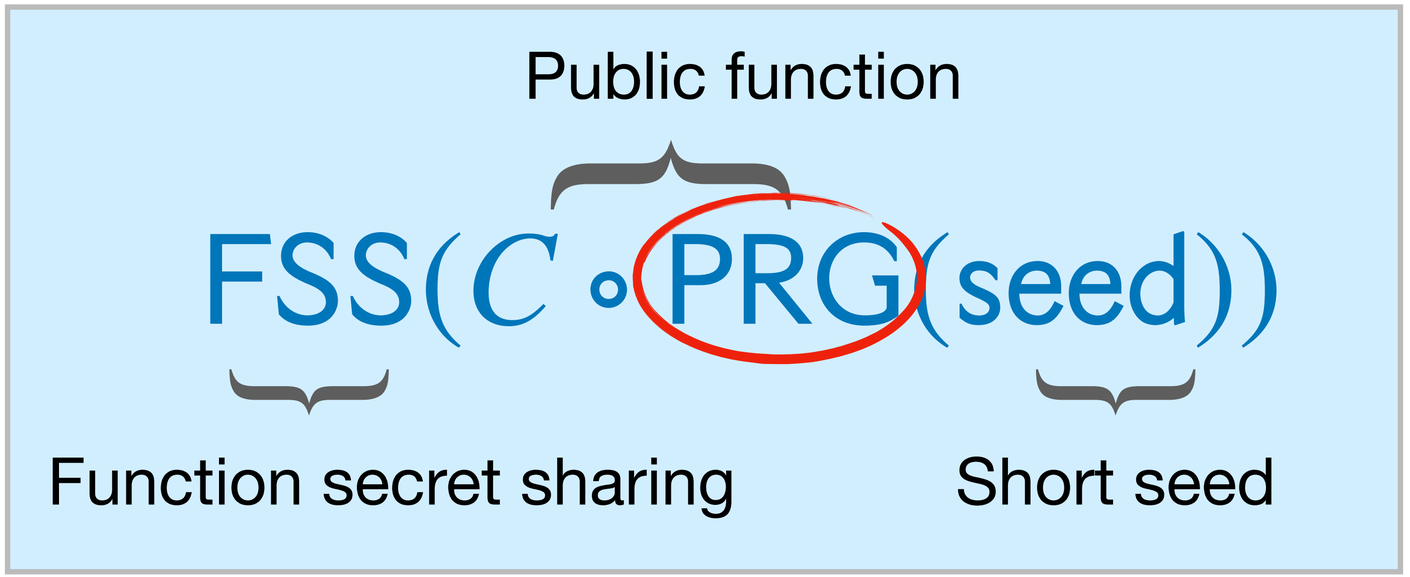
Giving Δ and sharing the b_i's are both essentially sharing a $\sqrt{N}$-size point function again: we can recurse the process!

This + later improvements [BGI16]:
FSS for point functions with keys of size
$O(\lambda \cdot \log N)$

Back to the PCG Template



We can succinctly share point functions

Back to the PCG Template

Public function

$$\mathrm{FSS}(C \circ \mathrm{PRG}(\mathrm{seed}))$$

Function secret sharing — Short seed

Secret sharing is additively homomorphic!

We can succinctly share point functions

Linear combinations of

Back to the PCG Template

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Are there any PRGs in this class?

LPN to the Rescue

The LPN assumption - primal

LPN to the Rescue

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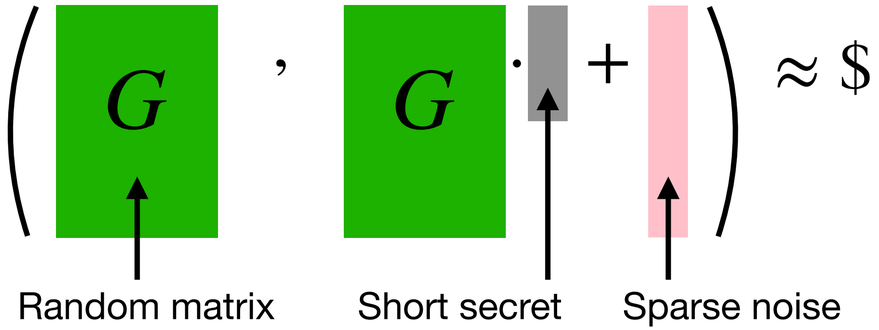
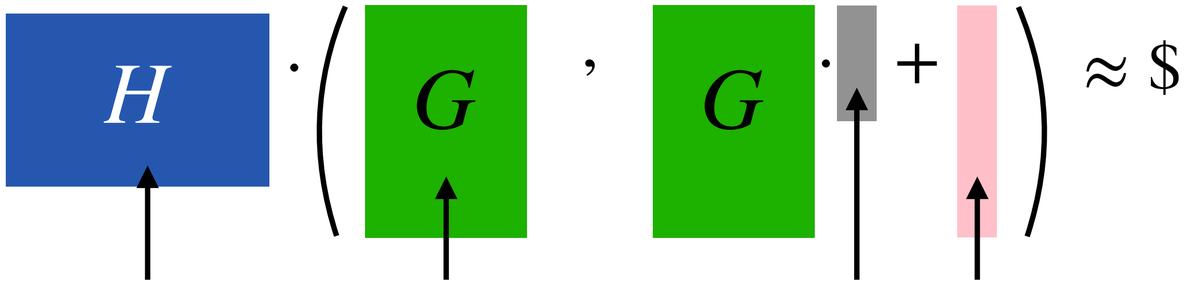


LPN to the Rescue

The LPN assumption - primal



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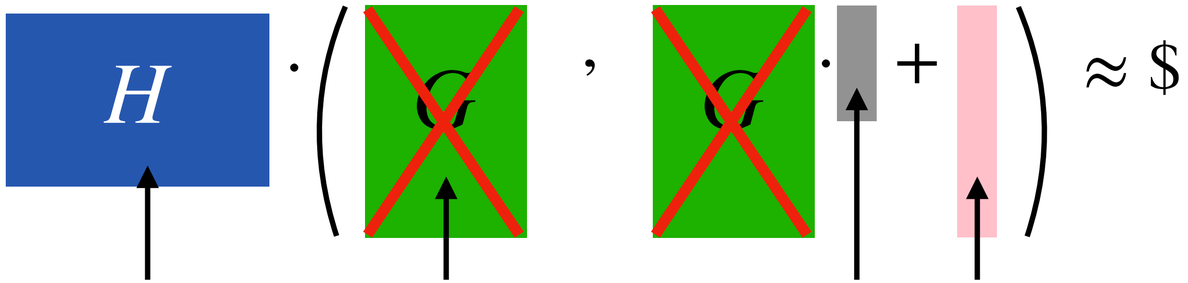
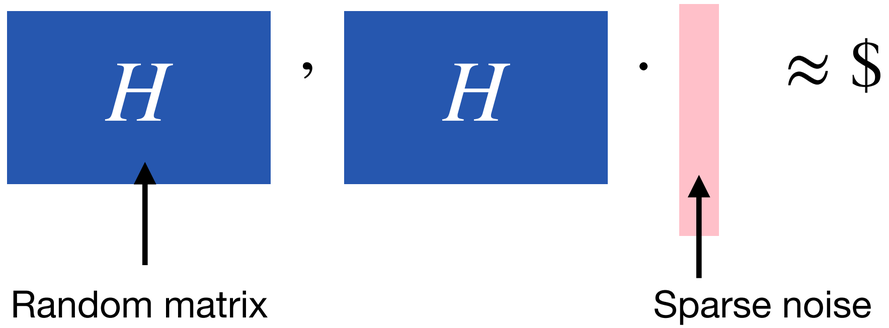


LPN to the Rescue

The LPN assumption - dual



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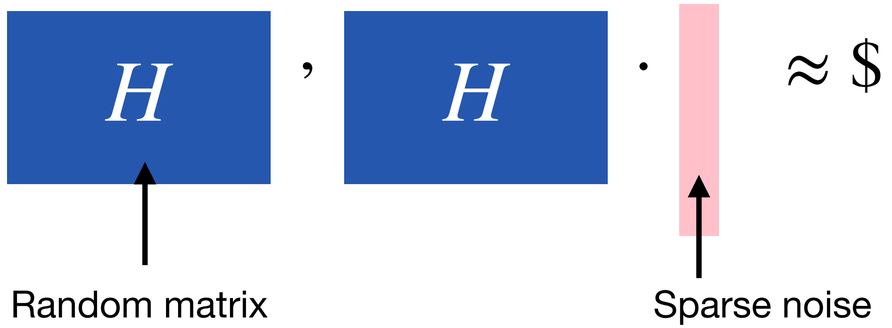
The LPN assumption - dual



LPN yields a simple PRG in the class:

LPN to the Rescue

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PRG : $(\alpha_i)_{i \le t} \mapsto H \cdot \sum_{i=1}^{t} \overrightarrow{u}_{\alpha_i}$, where $\overrightarrow{u}_{\alpha_i}$ is the unit vector with a 1 at α_i

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Linear combination

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LPN yields a simple PRG in the class:

$$\text{PRG} : (\alpha_i)_{i \le t} \mapsto \underbrace{H \cdot \sum_{i=1}^{t} \underbrace{\vec{u}_{\alpha_i}}_{\text{(Truth table of) point functions}}}_{\text{Linear combination}}$$

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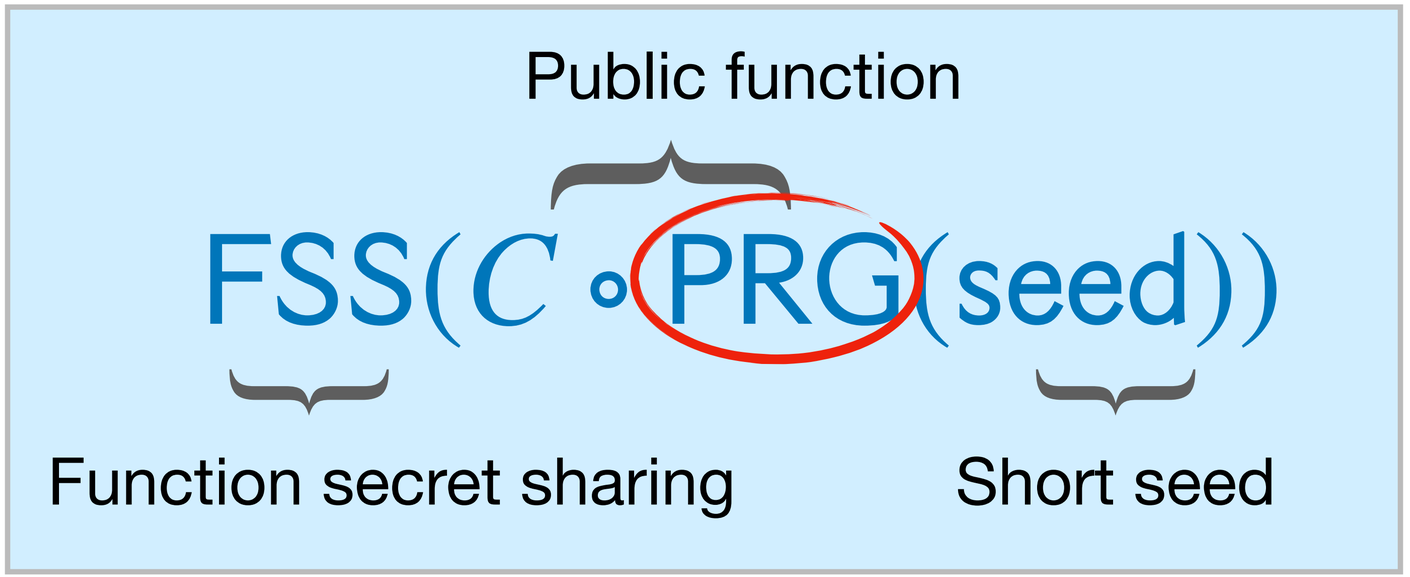
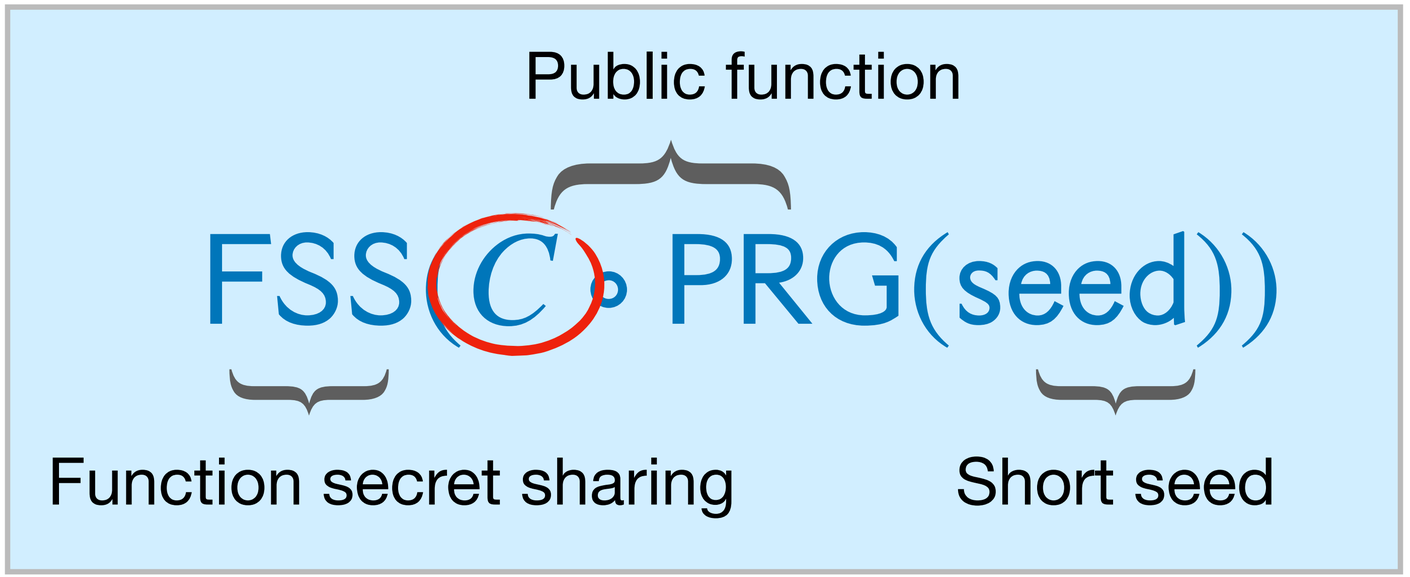
Back to the PCG Template Again



We have FSS for a class that contains a PRG

Back to the PCG Template Again



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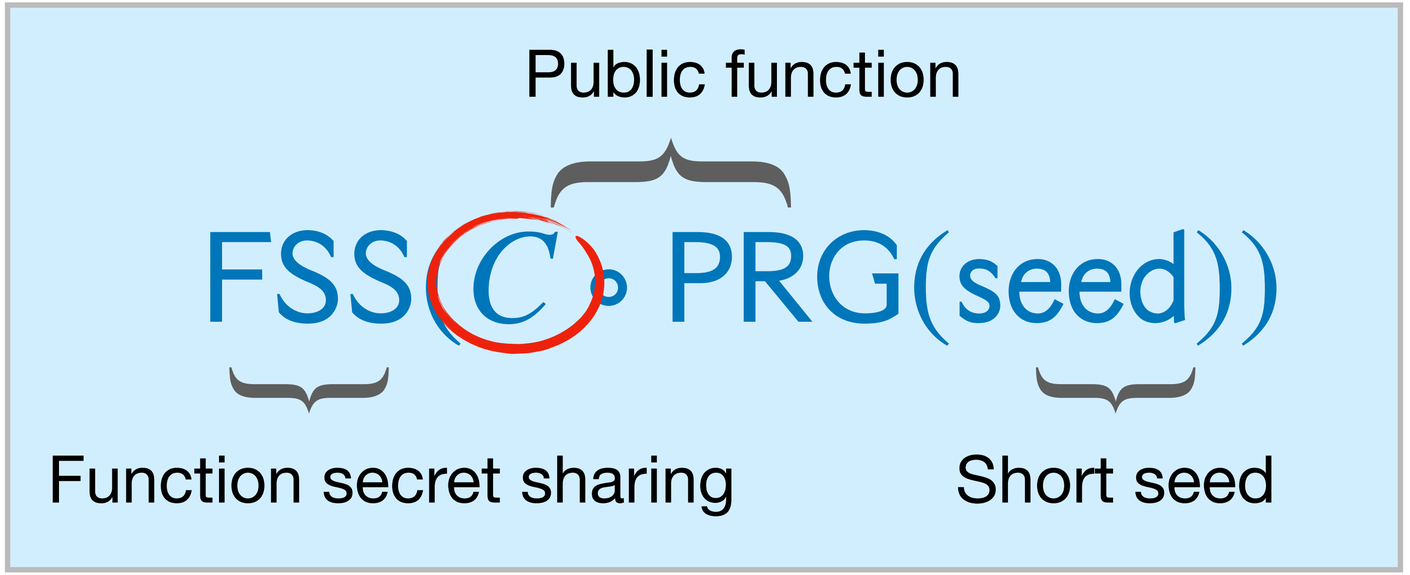
The heavy lifting in the many subsequent works boils down to:

- Making the PRG more efficient
- Adding support for more complex C

Back to the PCG Template Again



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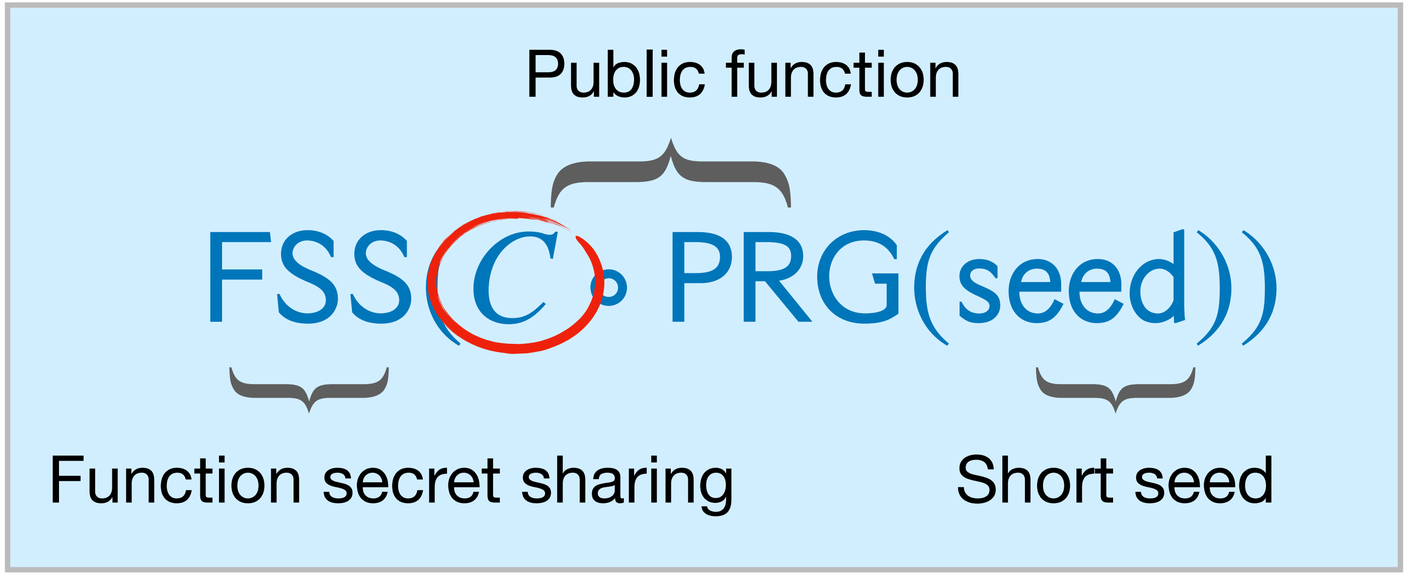

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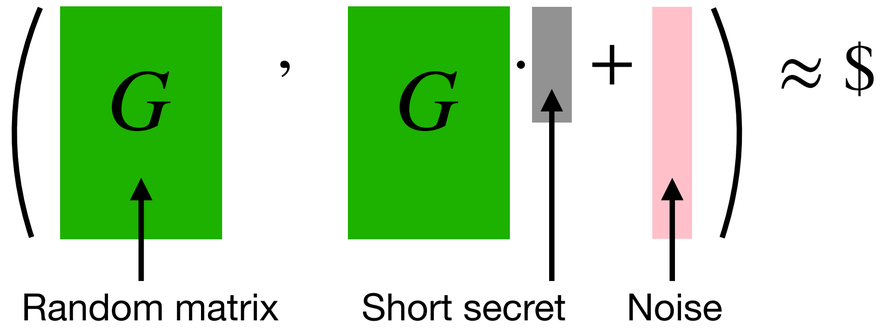
Digression: LPN versus LWE

LPN and LWE



LPN($\mathbb{F}_2$): $G \leftarrow_\$ \mathbb{F}_2^{m \times n}$, $\leftarrow_\$ \mathbb{F}_2^n$, $\mathrm{Ber}(\mathbb{F}_2)^n$

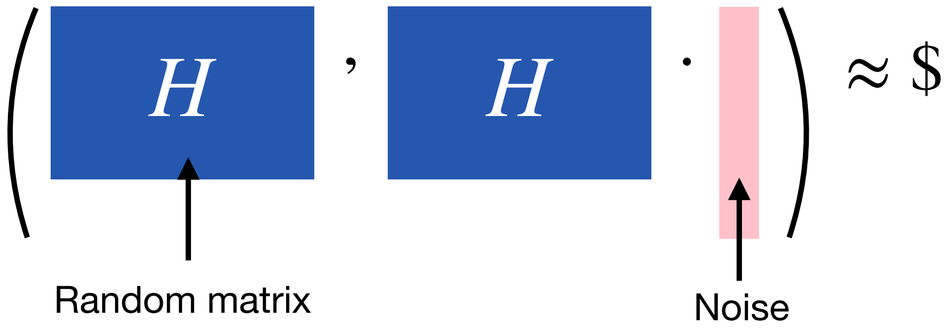
'Sparse'

LWE($\mathbb{F}_p$): $G \leftarrow_\$ \mathbb{F}_p^{m \times n}$, $\leftarrow_\$ \mathbb{F}_p^n$, $[-B, B]^n$

'Small'

Digression: LPN versus LWE

LPN and LWE



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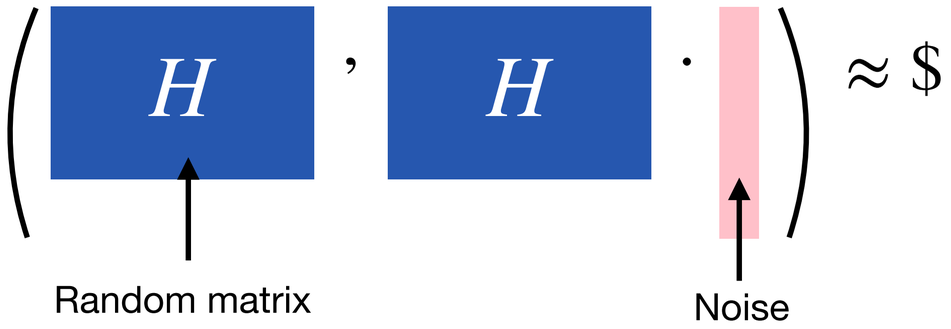

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Statistical security

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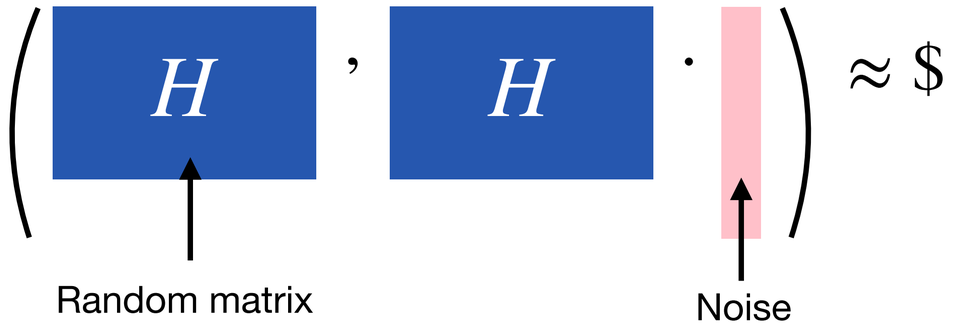
'Small'

$O(n)$ entropy in the noise $\Longrightarrow$ LHL, statistical security, lattice trapdoors, lossiness…

Digression: LPN versus LWE

LPN and LWE



Compressibility

LPN($\mathbb{F}_2$): $H \leftarrow_\$ \mathbb{F}_2^{m \times n}$, $\mathrm{Ber}(\mathbb{F}_2)^n$

'Sparse'

$t \cdot \log n \ll n$ entropy in the noise $\Longrightarrow$ compressibility! Crucially used in recent results: PCGs, but also iO and batch OT.

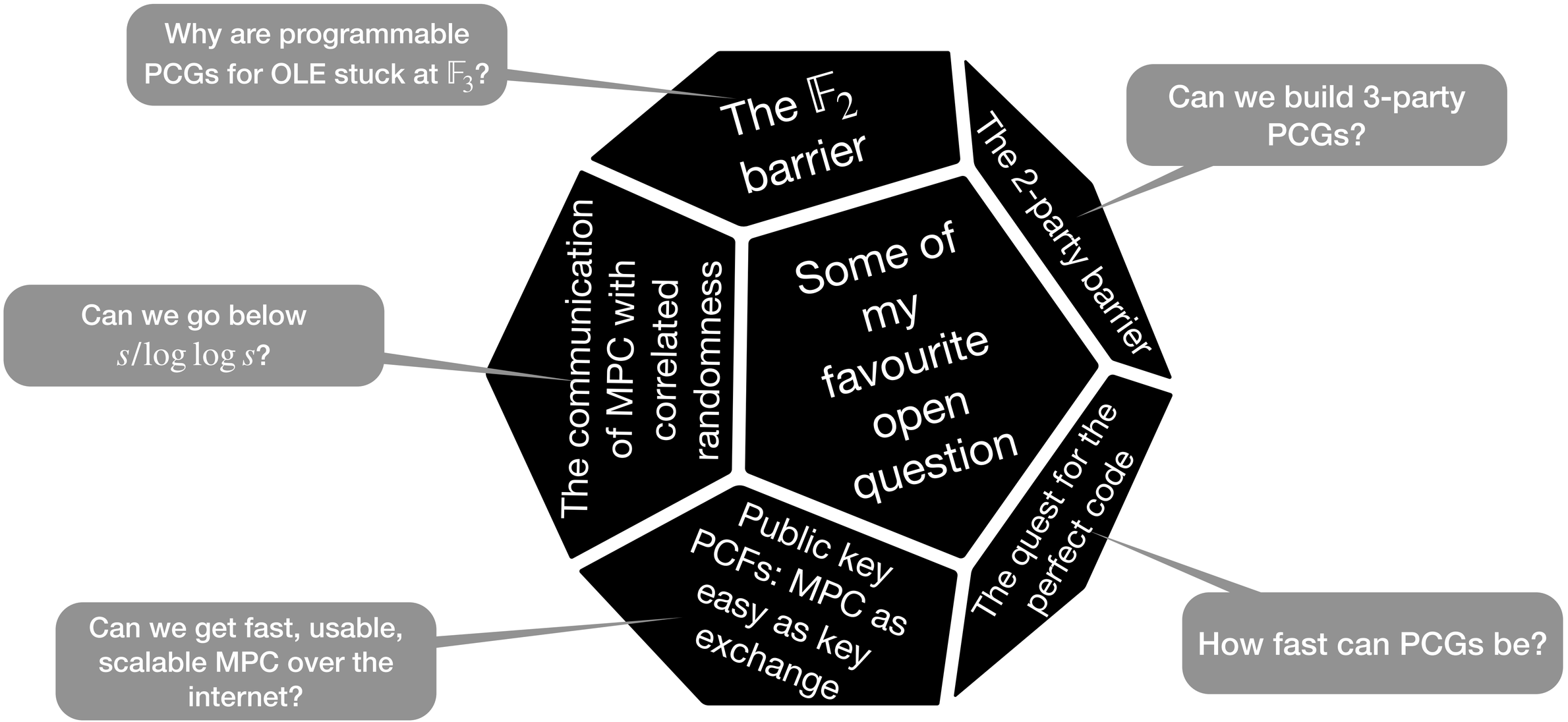
Statistical security

LWE($\mathbb{F}_p$): $H \leftarrow_\$ \mathbb{F}_p^{m \times n}$, $[-B, B]^n$

'Small'

$O(n)$ entropy in the noise $\Longrightarrow$ LHL, statistical security, lattice trapdoors, lossiness…

Some of my Favourite Open Questions



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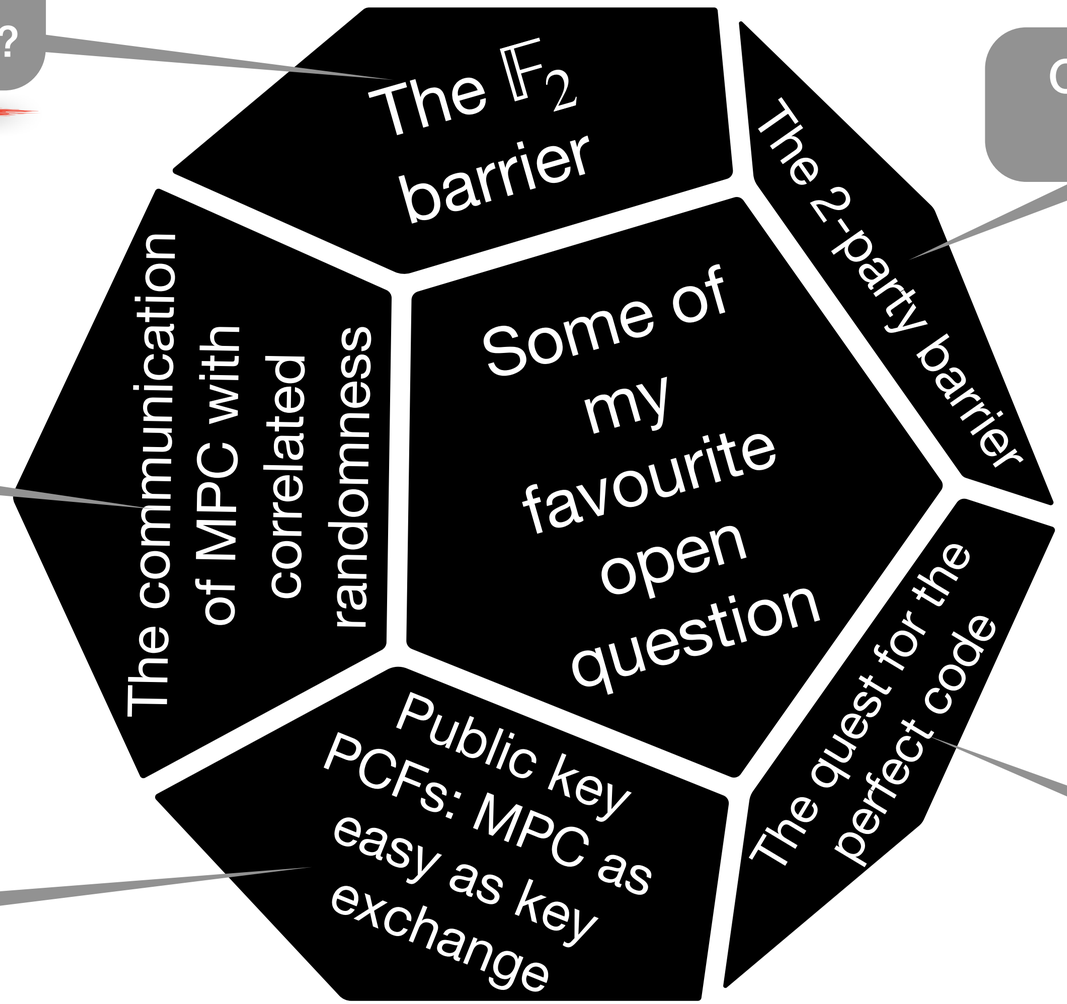


Why are programmable PCGs for OLE stuck at $\mathbb{F}_3$?

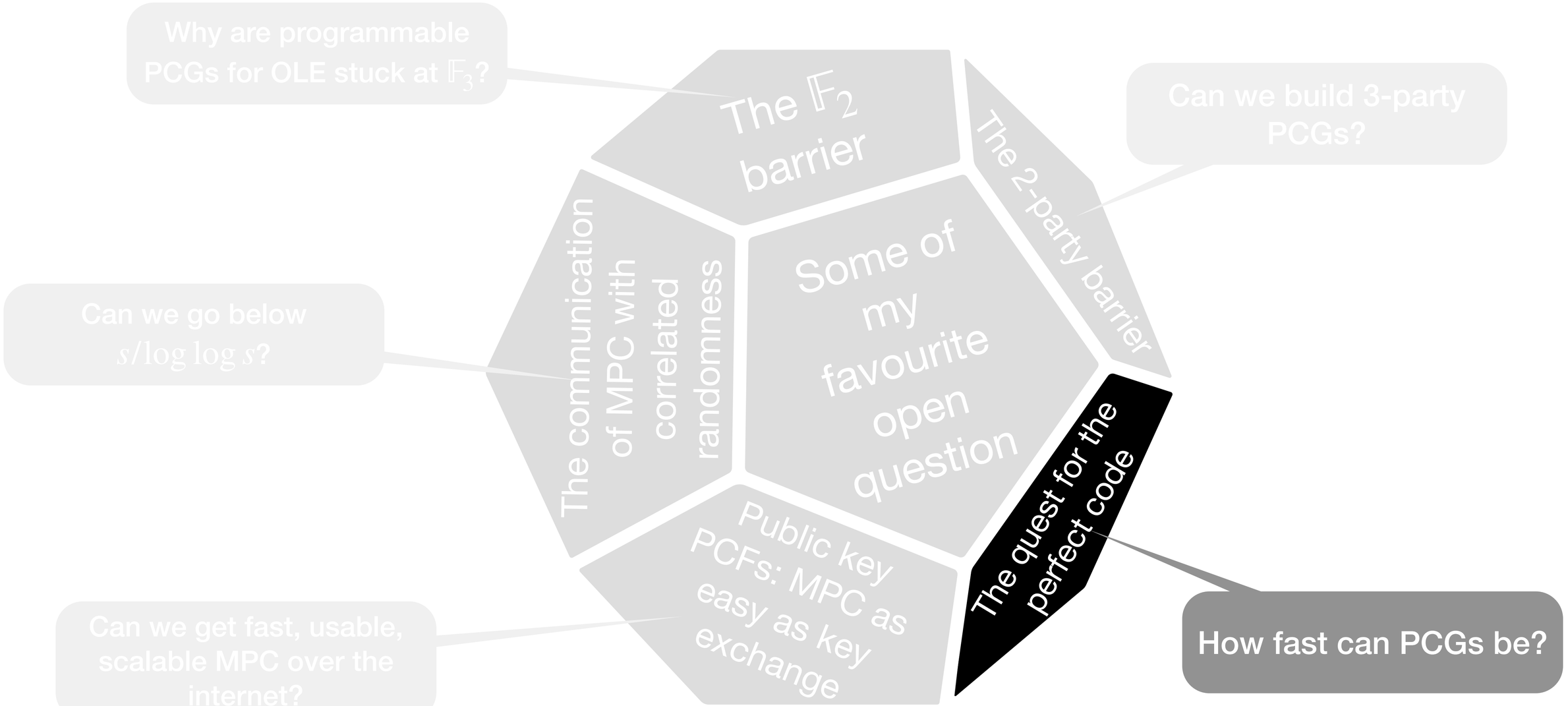
Can we build 3-party PCGs?

Can we go below $s/\log\log s$?

How fast can PCGs be?

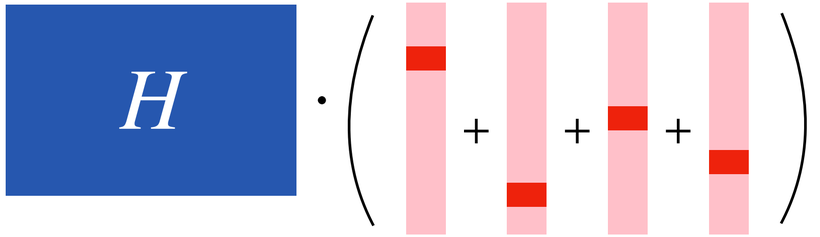
Can we get fast, usable, scalable MPC over the internet?

Some of my Favourite Open Questions



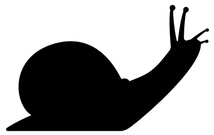
Making the PRG more Efficient

Making the PRG more Efficient

PRG : $(\alpha_i)_{i \leq t} \mapsto$ 

Making the PRG more Efficient

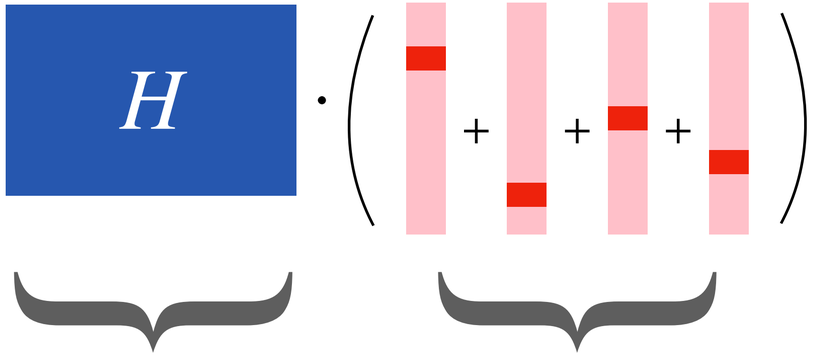
$$\text{PRG} : (\alpha_i)_{i \leq t} \mapsto H \cdot \left(+ + + \right)$$

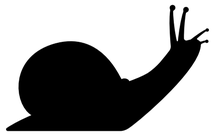


Multiplying by a random matrix of size $\Omega(n^2)$

Generating and summing unit vectors

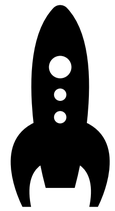
Making the PRG more Efficient

$$\text{PRG} : (\alpha_i)_{i \le t} \mapsto$$ 



Multiplying by a random matrix of size $\Omega(n^2)$

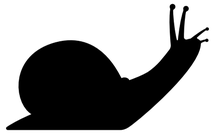
Generating and summing unit vectors



n is the total amount of correlated randomness we want to generate! (Think: $n \sim 2^{30}$)

Making the PRG more Efficient

$$\text{PRG} : (\alpha_i)_{i \le t} \mapsto H \cdot \left(+ + + \right)$$



Multiplying by a random matrix of size $\Omega(n^2)$

Generating and summing unit vectors

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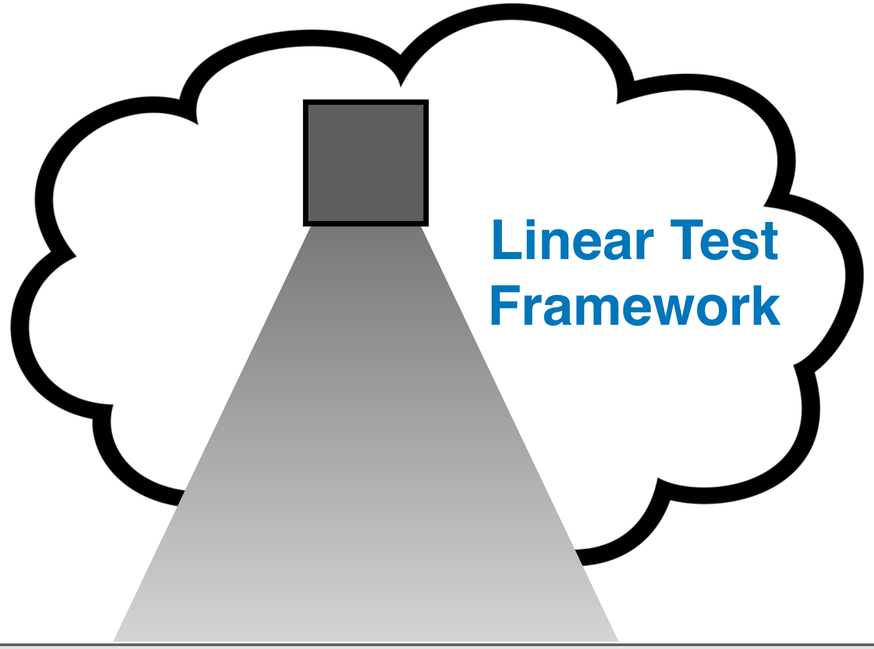
Can we replace H with a matrix that allows for fast matrix-vector product?

We need a rule of thumb to know which matrices will yield *plausible* variants of LPN

Security of (variants of) LPN - Linear Tests

A tremendous number of attacks on LPN have been published…



- **Gaussian Elimination attacks**
  - Standard gaussian elimination
  - Blum-Kalai-Wasserman [J.ACM:BKW03]
  - Sample-efficient BKW [A-R:Lyu05]
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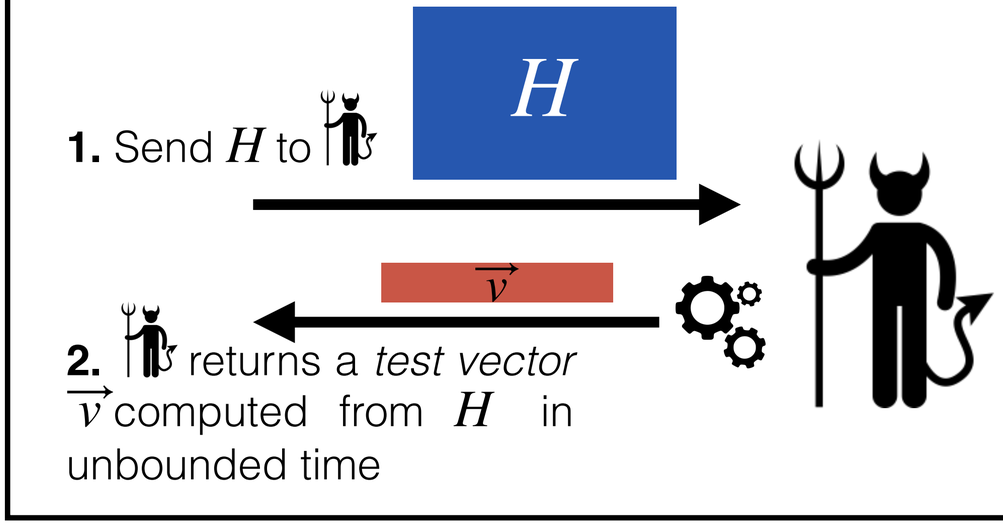
Crucial observation: most attacks fit in the same framework, the *linear test framework. (*)*

Game



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The adversary wins in the distribution induced by

$$\vec{v} \cdot \left(H \cdot \right)$$

(over a random choice of secret and sparse noise) is non-negligibly *biased*.

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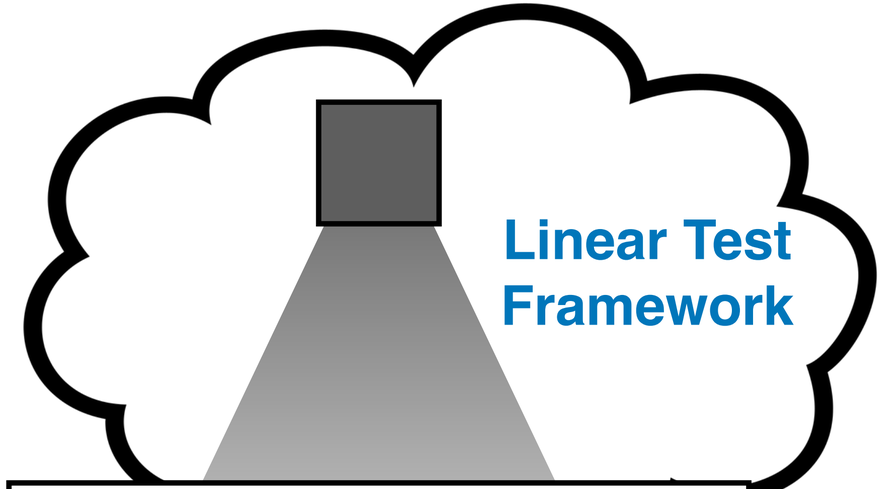

(*): highly structured algebraic codes (e.g. Reed-Solomon) are a different beast

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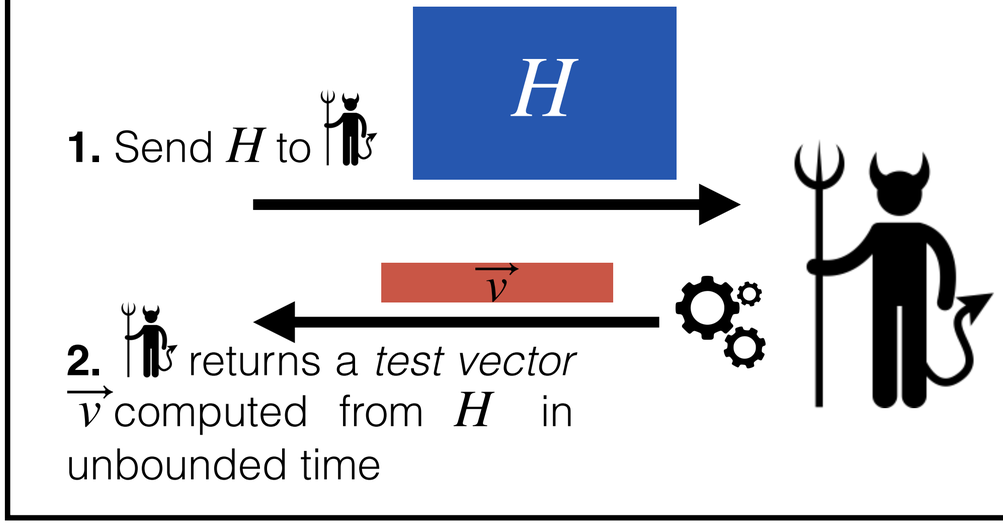

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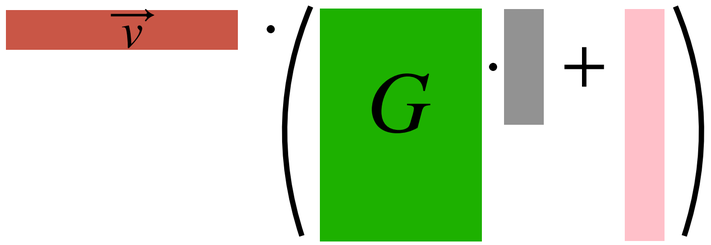

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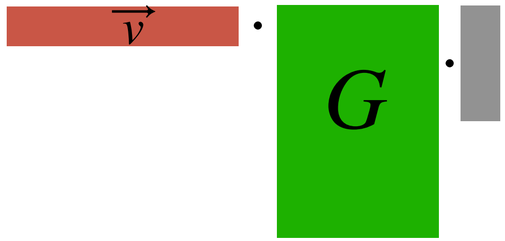
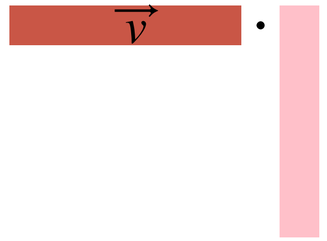
We have a sum of two distributions:

Induced by the *codeword*



Protects against *light* linear tests

Induced by the *noise vector*



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Claim: Assume t (number of noisy coordinates) is set to a security parameter. If there is a constant c such that every subset of $c \cdot n$ rows of G is linearly independent, no linear test can distinguish $G \cdot \vec{s} + \vec{e}$ from random.

Withstanding Linear Tests

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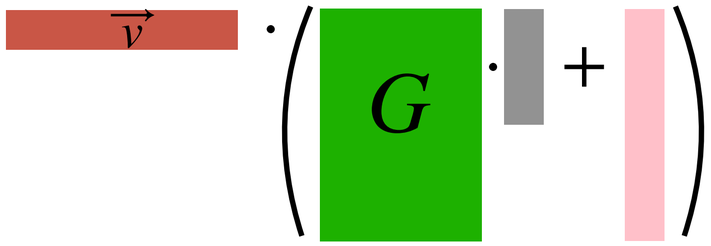

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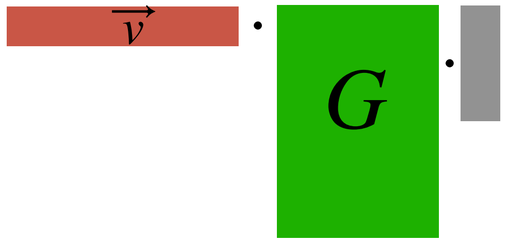

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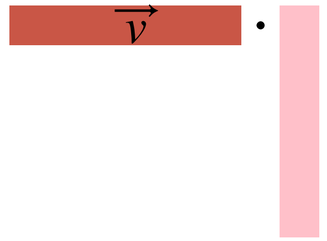

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Rephrasing the sufficient condition:

Every subset of $O(n)$ rows of G is linearly independent
$\iff$ the left-kernel of G does not contain nonzero vector of weight less than $O(n)$
$\iff$ the *dual code* of G, i.e., the code generated by *the transpose of its parity check matrix* H, has linear minimum distance

Pseudorandom Correlation Generators - Efficiency

Goal: computing  fast, such that the code  is LPN-friendy

We want to find a matrix $M = H^{\intercal}$ such that (1) the code generated by M is a good code, and (2) computing ~~$M^{\intercal} \cdot \vec{v}$~~ $M \cdot \vec{v}$ takes time $O(n)$ for any $\vec{v}$ (this is the *transposition principle*)

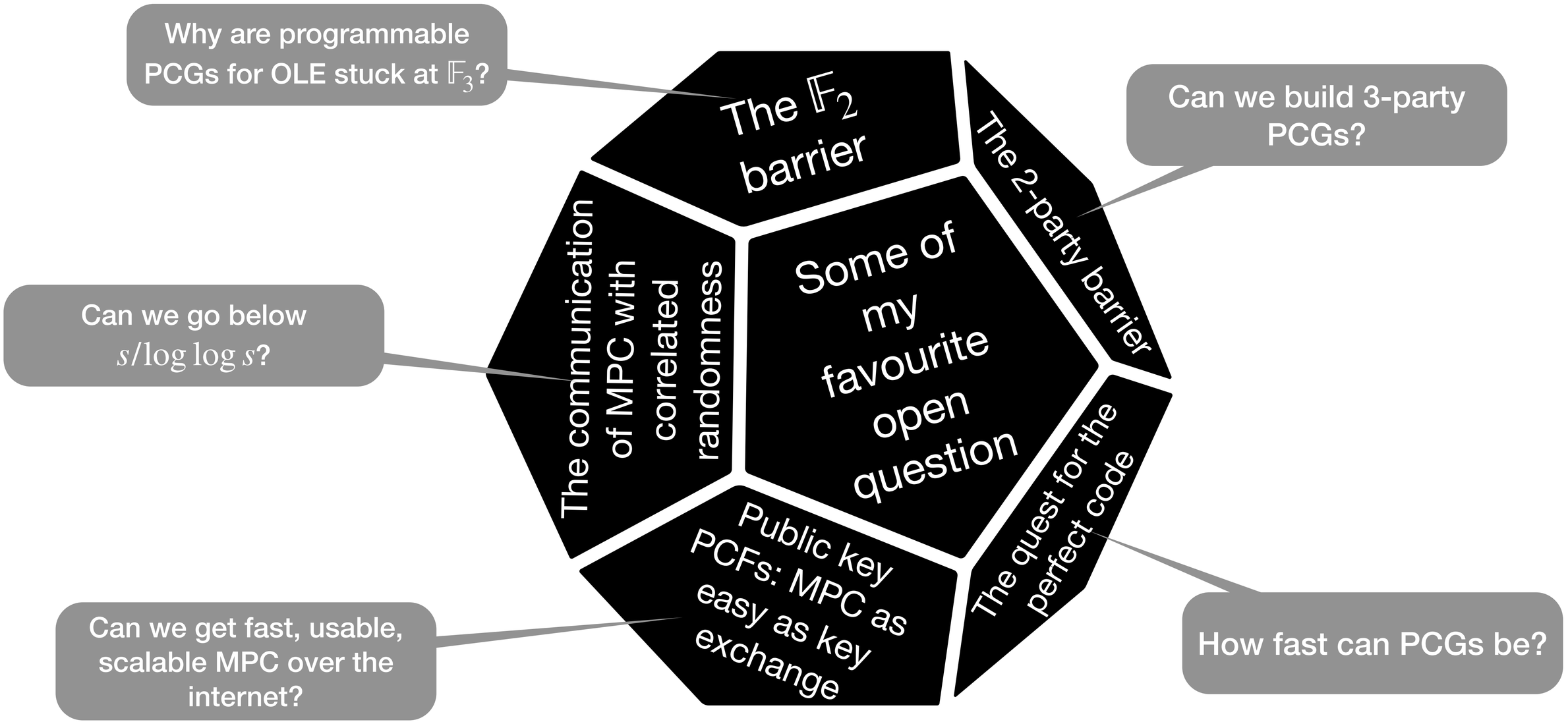
$\Longrightarrow$ We need to find a *good* and *linear-time encodable* code. And we want it concretely efficient!

Pseudorandom Correlation Generators - Efficiency

There is an ongoing and exciting quest for pinpointing the *right* code for PCG applications:

- CCS:Boyle-C-Gilboa-Ishai'18 suggested using LDPC code
- CCS:Boyle-C-Gilboa-Ishai-Kohl-Rindal-Scholl'19 moved to quasi-cyclic codes
  due to concern regarding linear-time encoding of LDPC codes
- Crypto:C-Raghuraman-Rindal'21: tailored LDPC with heuristic & experimental support
- Crypto:Boyle-C-Gilboa-Ishai-Kohl-Resch—Scholl'22: Expand-Accumulate codes
- Latest news: there's apparently a new proposal that suggests Expand-Convolute codes instead (and which breaks Silver along the way!)
- There are a few more codes I'd like to investigate, the quest continues!

Some of my Favourite Open Questions



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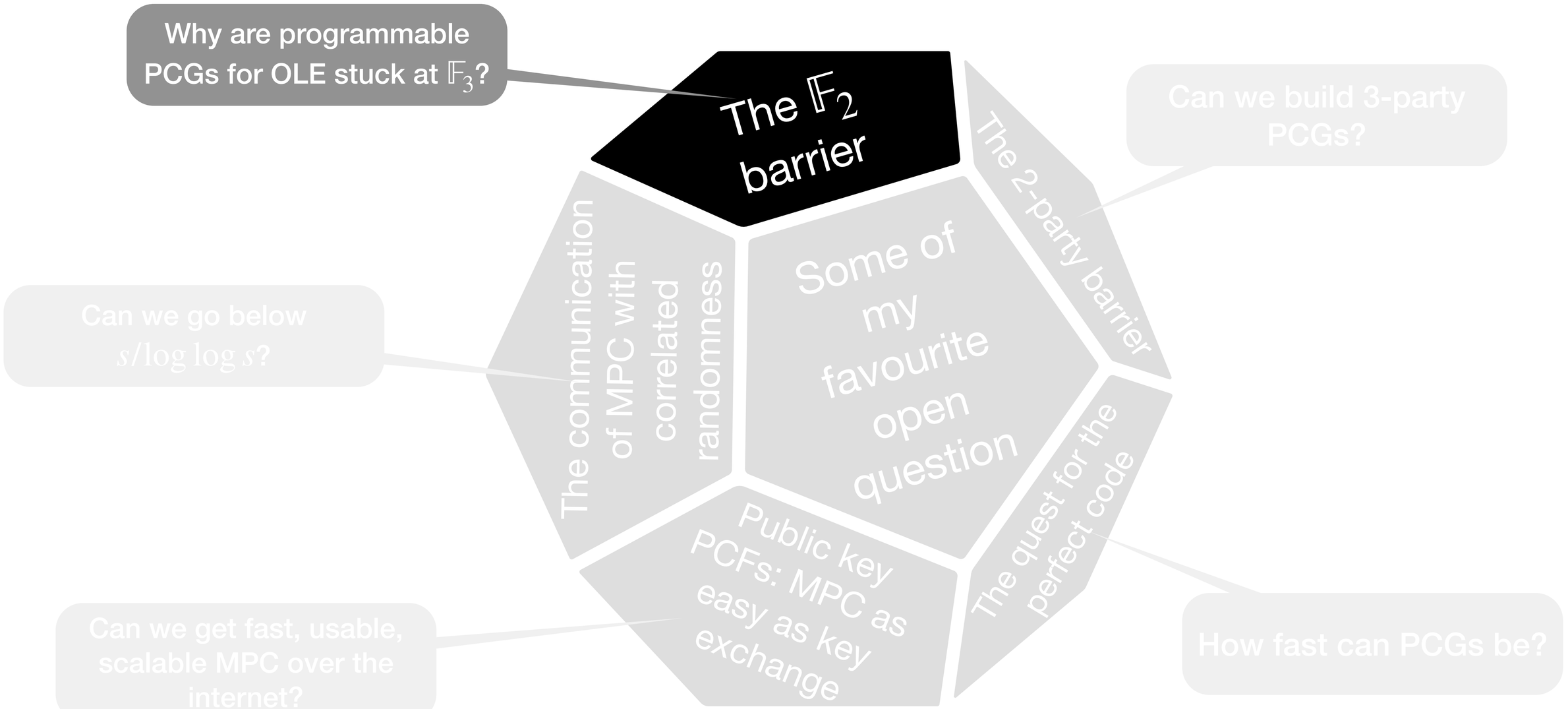


OLE Correlations

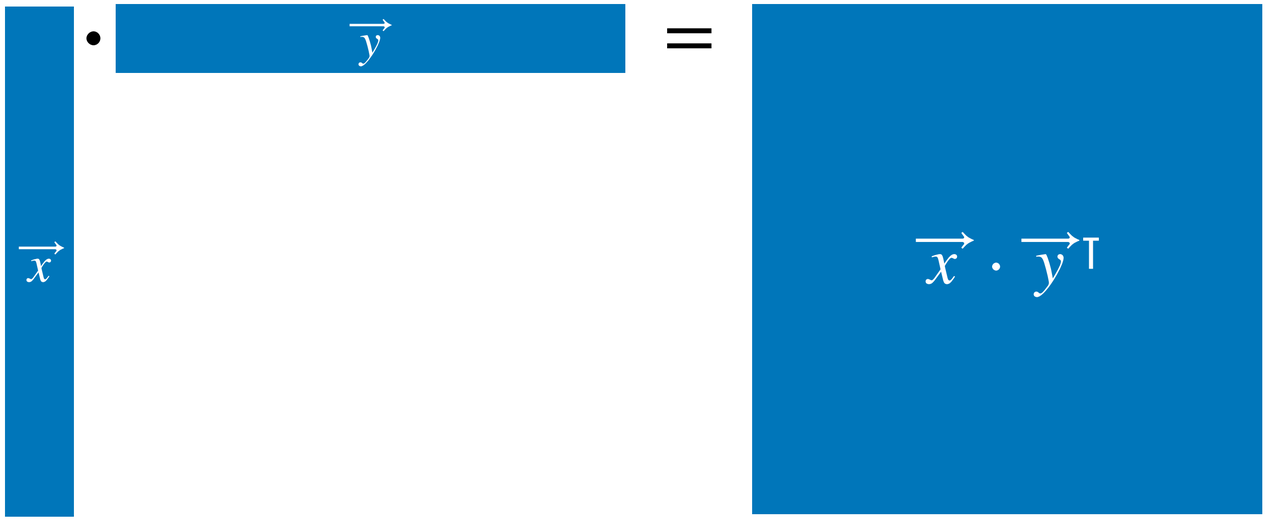
OLE over $\mathbb{F}$ is the type of correlation we want to do (semi-honest) secure computation of arithmetic circuits over $\mathbb{F}$.

In an OLE, Alice gets $a \leftarrow \mathbb{F}$, Bob gets $b \leftarrow \mathbb{F}$, and Alice and Bob get random shares of $a \cdot b$.

OLE Correlations, the LPN Way

Goal:

- Alice gets a pseudorandom vector $\vec{x}$
- Bob gets a pseudorandom vector $\vec{y}$
- Alice and Bob get shares of $\vec{x} \odot \vec{y}$



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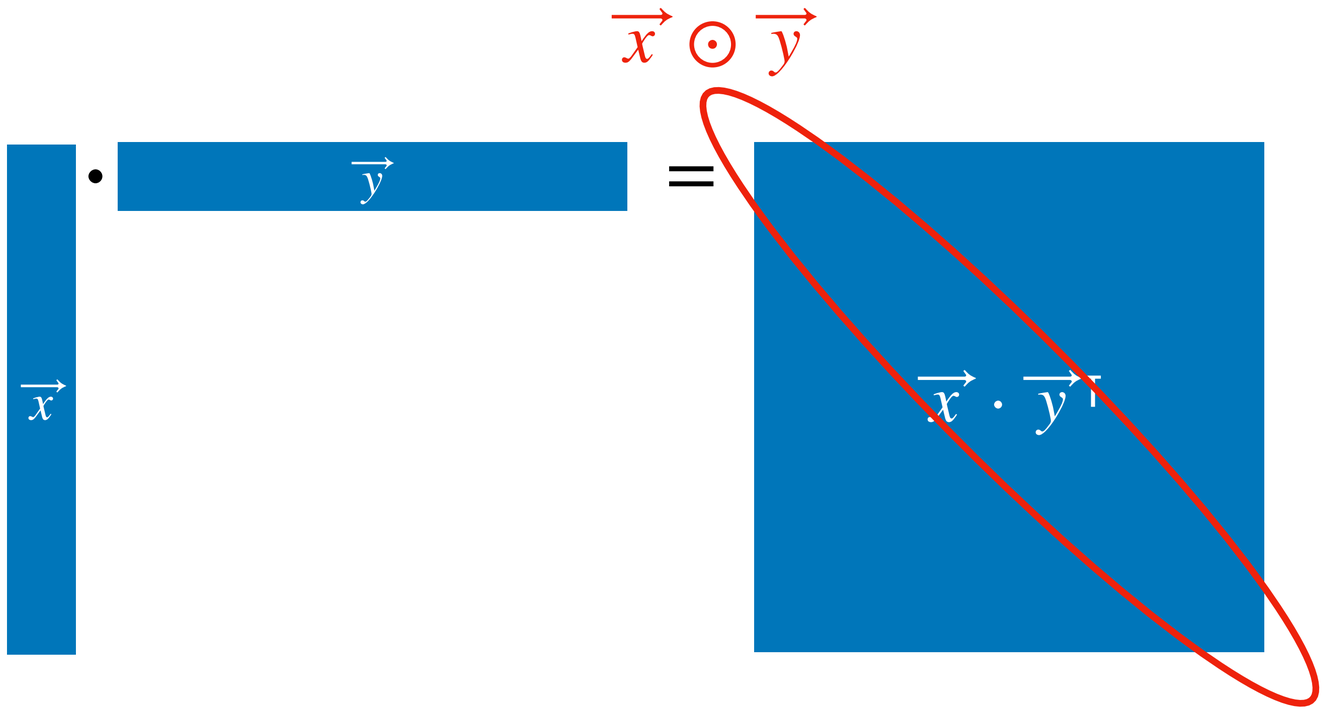

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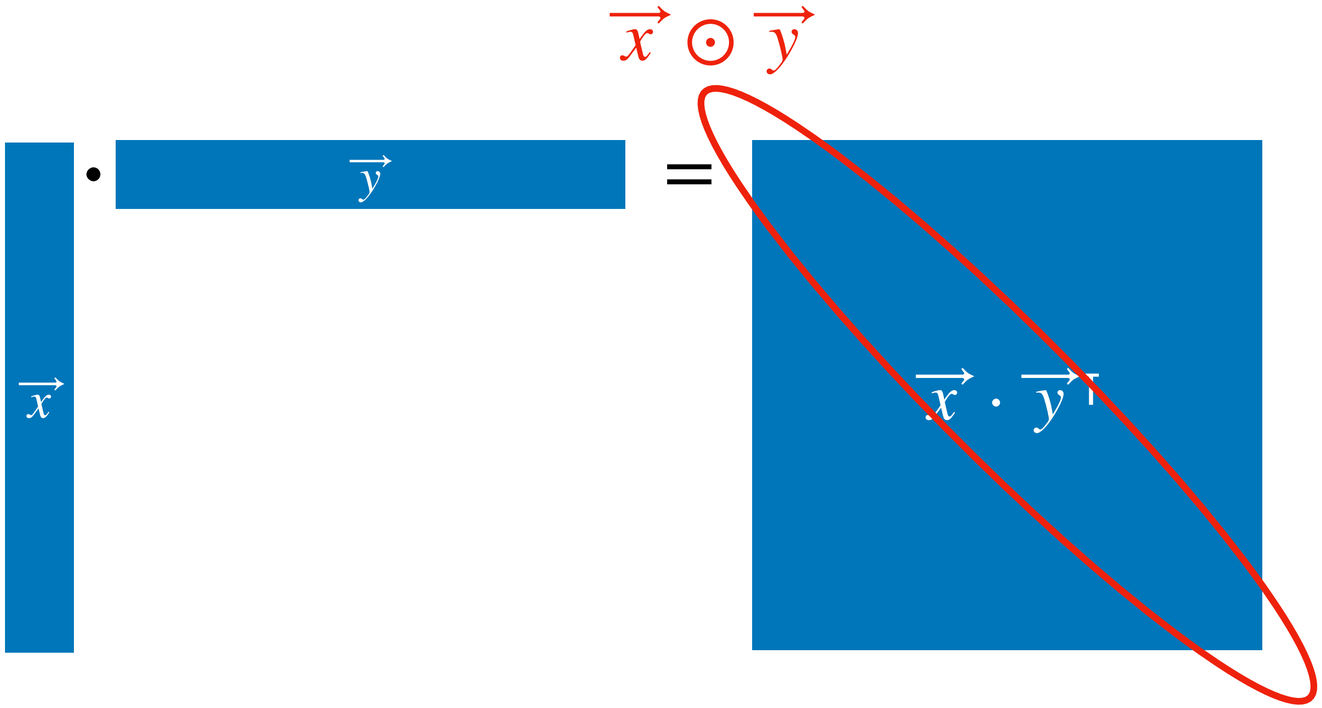

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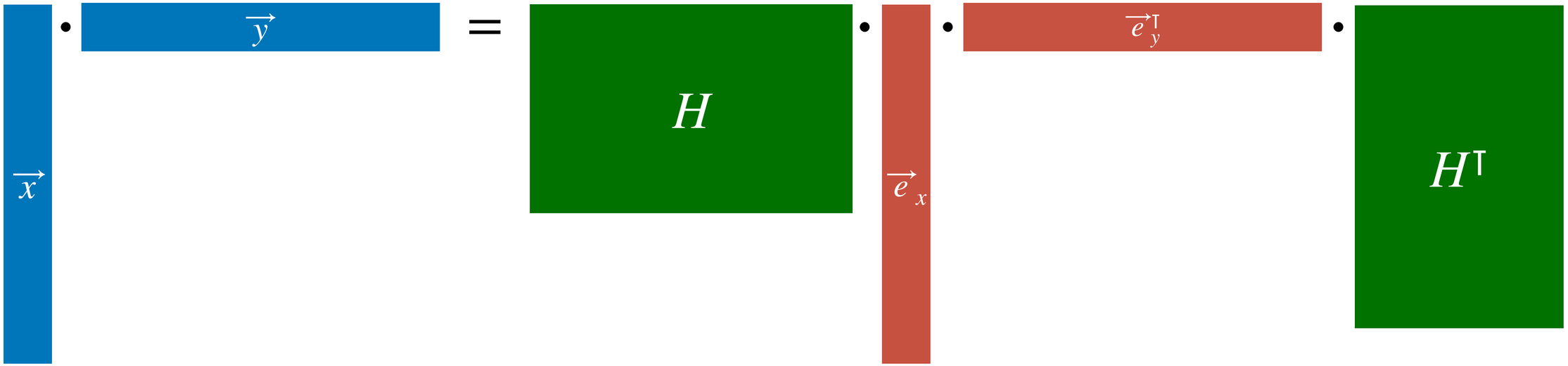

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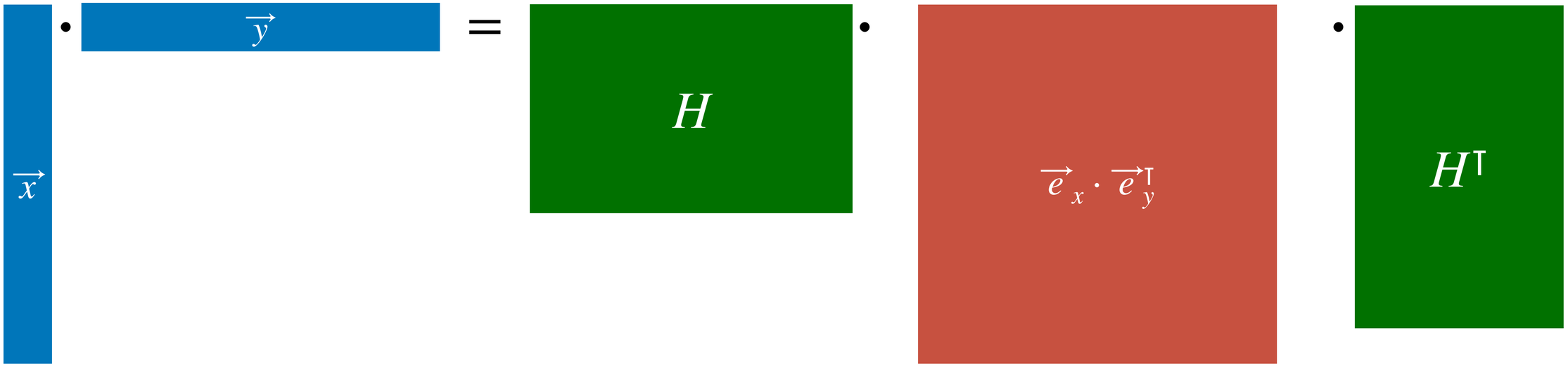

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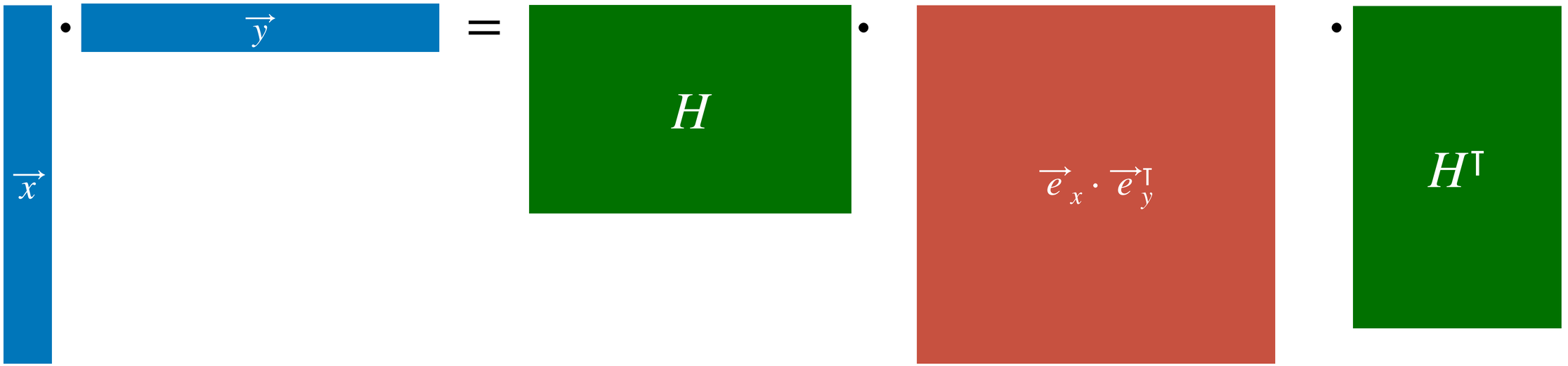
$\Longrightarrow$ can be generated with comm. $O(\lambda t^2 \log n)$



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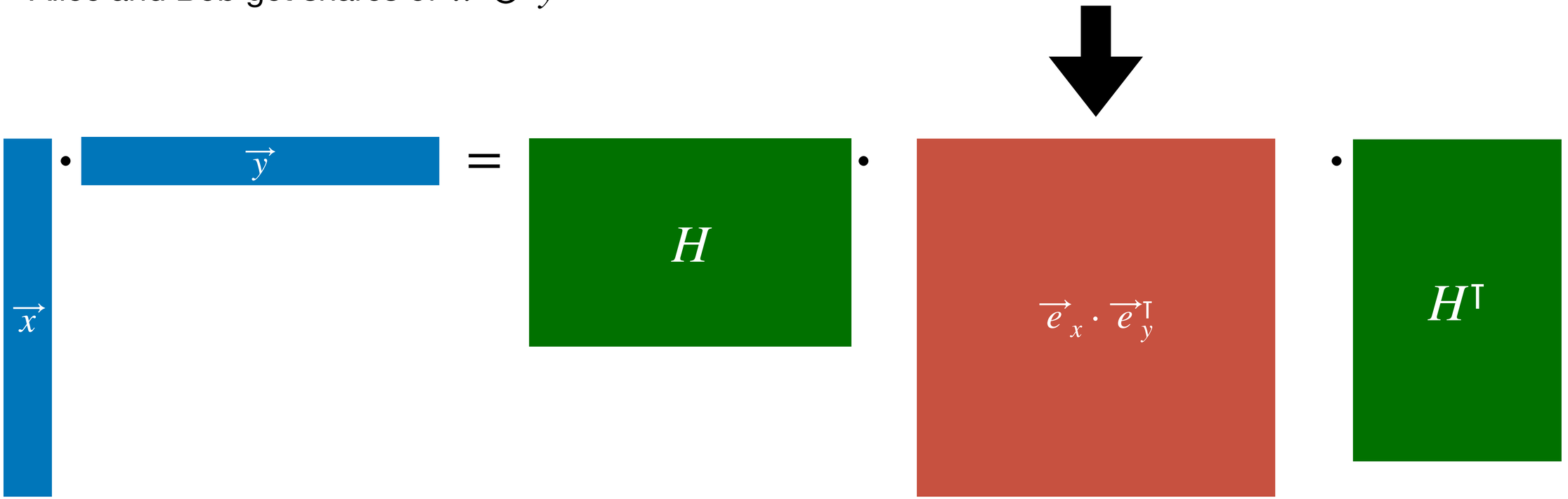

Uses only LPN

Costs $\omega(n^2)$! (Think: $n \sim 2^{30}\ldots$)

OLE Correlations, the *Ring*-LPN Way

Crypto: Boyle-C-Gilboa-Ishai-Kohl-Scholl'20

Let $\mathscr{R}$ be the ring $\mathbb{Z}_p/F(X)$ where $F(X)$ is a degree-n polynomial that splits entirely, and $p > n$.

Ring-LPN assumption: $(a, b) \sim (a, a \cdot e + f)$ where $(a, b) \leftarrow \mathscr{R}$ and (e, f) are random *t-sparse* polynomials.

Observation: we can get n OLE correlations from a single 'ring-OLE' correlation $(x, y, \langle x \cdot y \rangle)$ over $\mathscr{R}$: the OLE correlations are obtained by reducing x, y, and $x \cdot y$ modulo each of the linear factors F_i of F.

Construction:

- Alice gets a pseudorandom polynomial $x = a \cdot e_x + f_x$ where (e_x, f_x) are t-sparse polynomials over $\mathscr{R}$
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- 'Splittable ring-LPN' deserves further study
- $\mathbb{F}$ must be large!

OLE Correlations, from Quasi-Abelian Syndrome Decoding

How do we break this ‘field-size barrier’? An answer in our recent Crypto paper (Bombar-C-Couvreur-Ducros’23): we move to *quasi-abelian* codes, which are defined over group algebras.

High level intuition

The group algebra structure gives a suitable framework to find the *right* polynomial P to instantiate an LPN variant over a ring $\mathscr{R} = \mathbb{F}[X_1, \cdots, X_d]/P(X_1, \cdots, X_d)$ such that

- $\mathscr{R} \sim \mathbb{F} \times \cdots \times \mathbb{F}$ (*i.e.* we get many copies of an OLE over $\mathbb{F}$)
- The underlying assumption is plausibly secure (*i.e.* resists linear attacks)

Using multivariate rings gives us many more roots of P even for a small $\mathbb{F}$! In fact, we can get up to $(|\mathbb{F}| - 1)^d$ copies of an OLE over $\mathbb{F}$.

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This only gives something meaningful up to $\mathbb{F}_3$!

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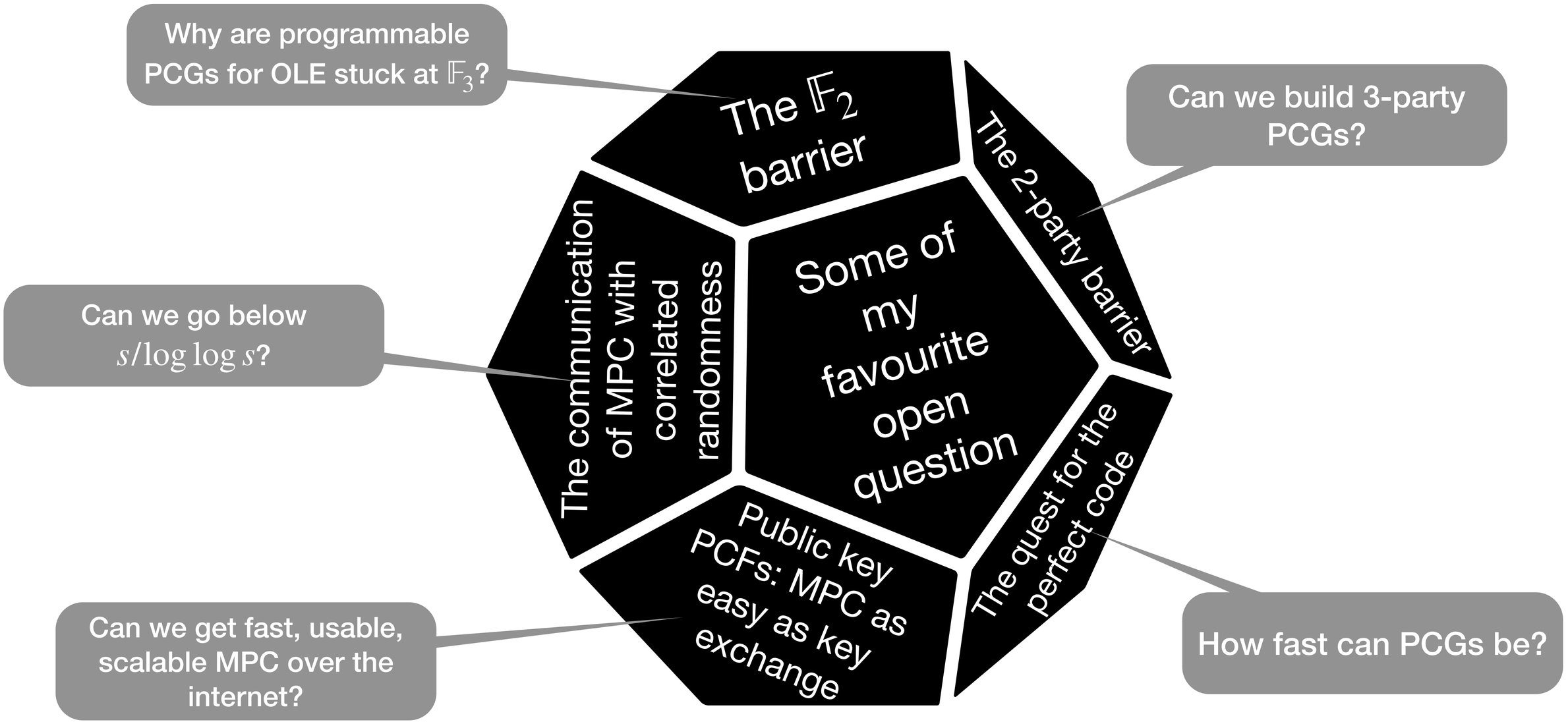

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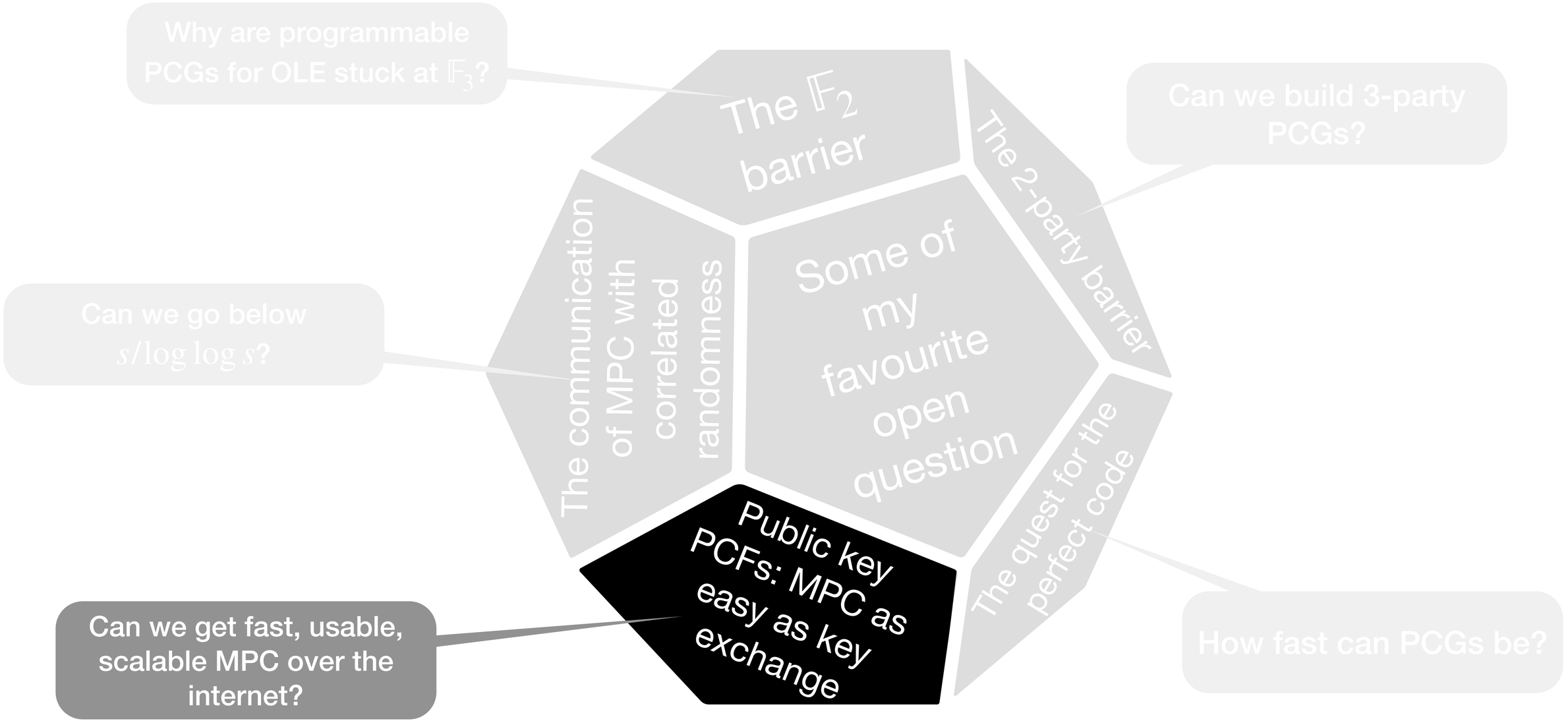


A Closer Look at Secure Communication

Our ultimate goal is *practical* MPC that can be deployed and used over the web

Secure communication is already widely deployed and in use

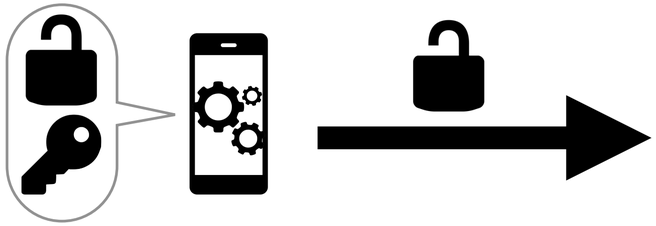
> 85% of the total internet traffic is encrypted

$\Longrightarrow$ Let us look at secure communication's recipe for success!

A Closer Look at Secure Communication

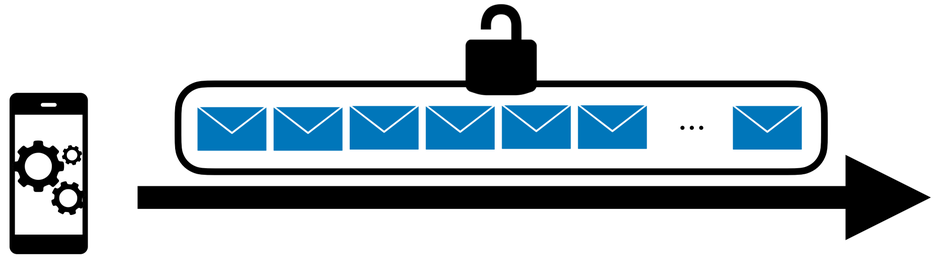
Two Phases:

Key exchange phase



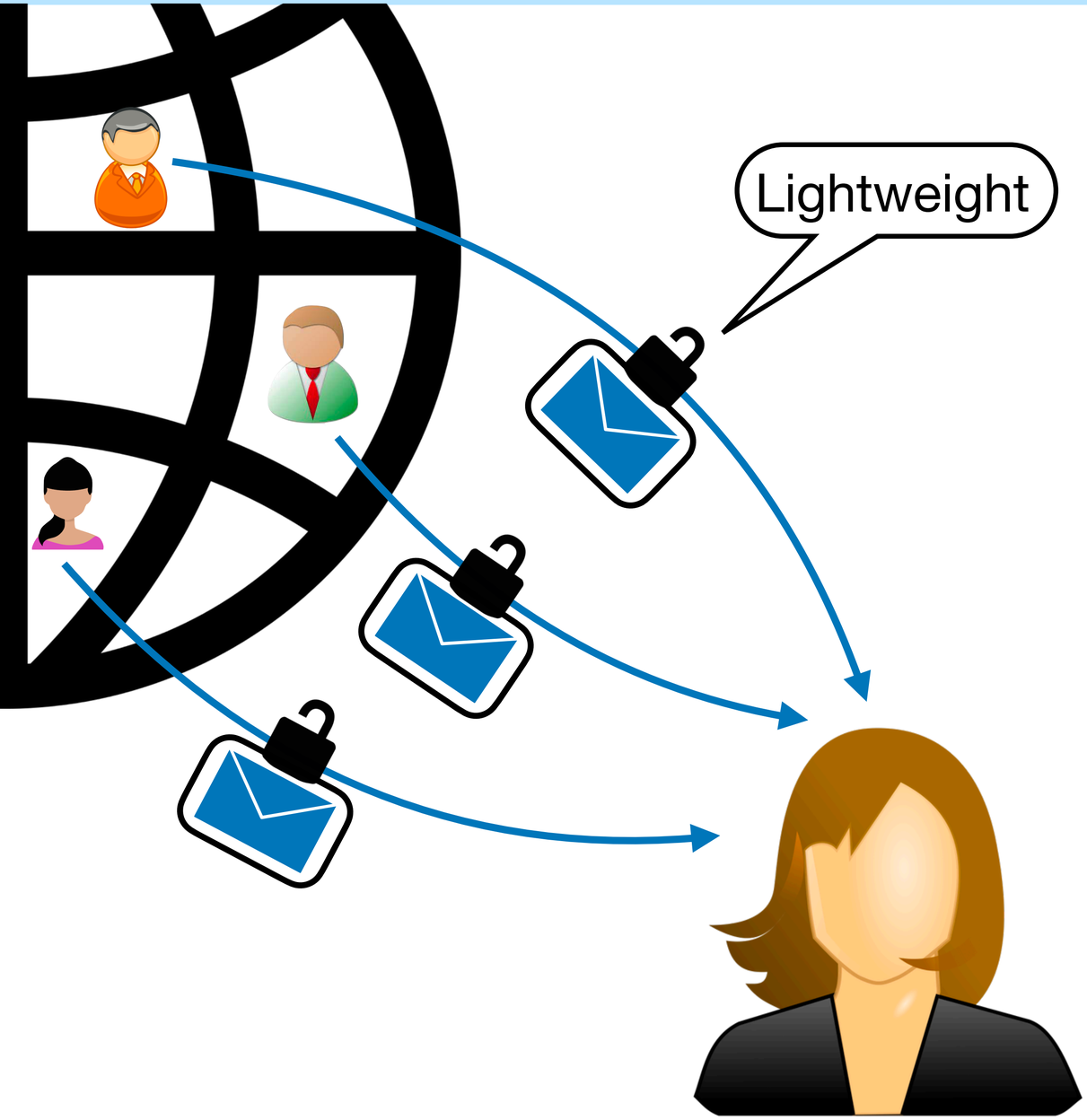
- One-time, *simultaneous* interaction
- Heavy (public key) computations
- Low communication $n \cdot |\text{🔒}|$ (not n^2)

Encryption phase



- Lightweight (symmetric) computations
- Optimal message-to-cipher ratio

A Closer Look at Secure Communication



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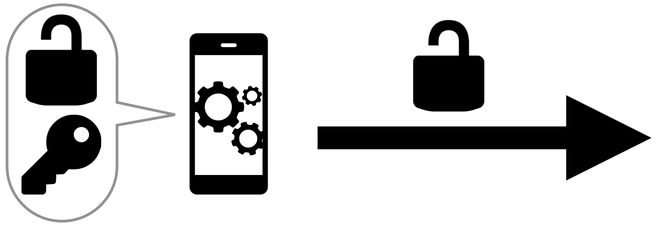

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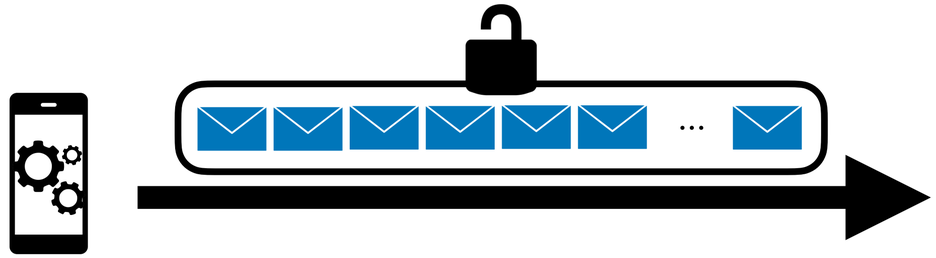

- Lightweight (symmetric) computations
- Optimal message-to-cipher ratio

Back to the PCG Template Again

Public function

$$\mathrm{FSS}(C \circ \mathrm{PRG}(\mathrm{seed}))$$

Function secret sharing

Short seed

Using a PRG enables a *one-time* generation of a *fixed* amount of correlations

Back to the PCG Template Again

Public function

$$\mathrm{FSS}(C \circ \mathrm{PRF}(\mathrm{seed}))$$

Function secret sharing

Short seed

A pseudorandom correlation *function* is to a PCG what a PRF is to a PRG

Back to the PCG Template Again

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Are there any FSS-friendly PRFs?

Back to the PCG Template Again

Public function

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Function secret sharing — Short seed

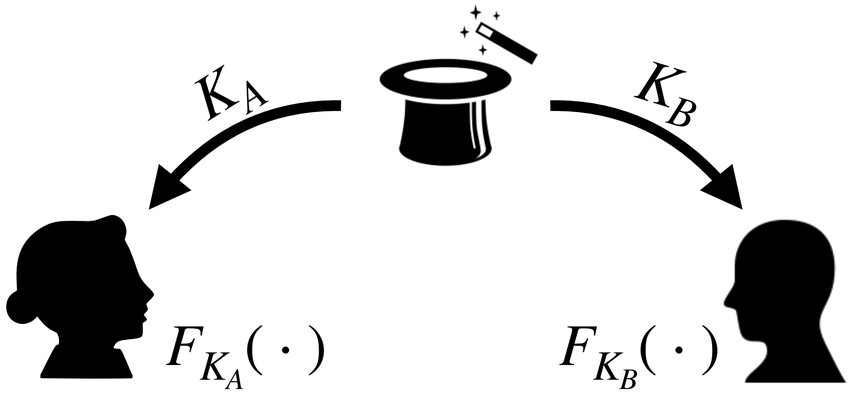
A pseudorandom correlation *function* is to a PCG what a PRF is to a PRG

Are there any FSS-friendly PRFs?

The existence of PRFs in low complexity classes yields strong limitations for learning theory

FOCS:BCGIKS20 and Crypto:BCGIKRS22 give plausible candidates

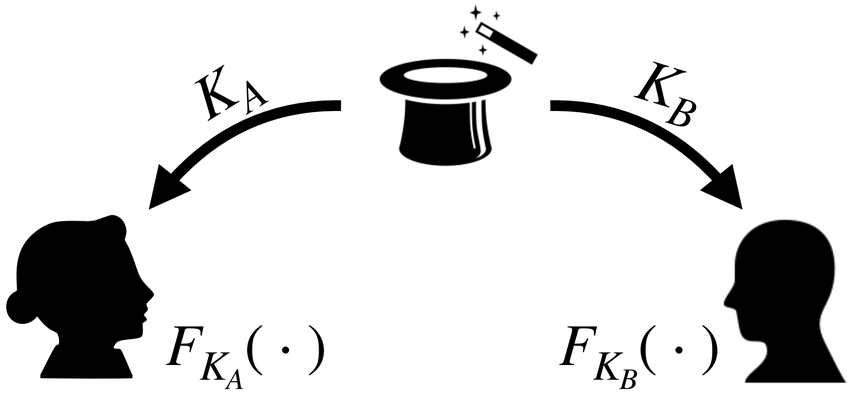
Pseudorandom Correlation Functions



Correctness & security:

- Black-box access to samples of the form $(F_{K_A}(x), F_{K_B}(x))$ are indistinguishable from black-box access to random samples from a target correlation.
- From the viewpoint of Alice, each $F_{K_B}(x)$ is indistinguishable from a random value sampled *conditioned on satisfying the correlation with* $F_{K_A}(x)$.
- Same condition in the other direction.

Public-Key Pseudorandom Correlation Functions



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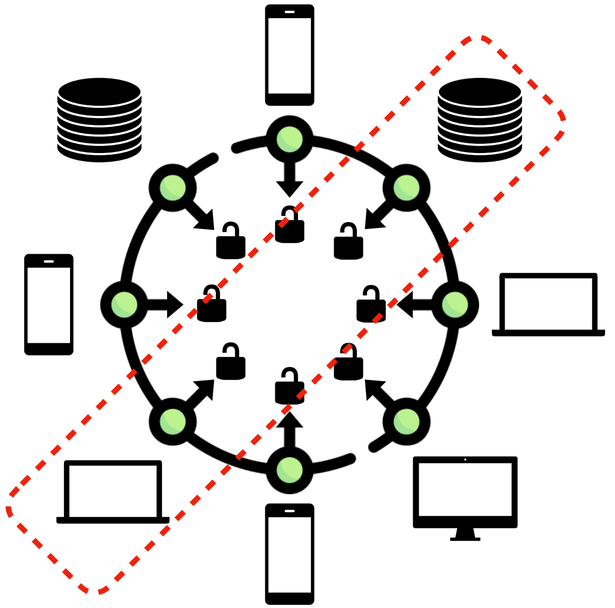
Achieving *non-interactive* silent key generation



Formally:

- KeyGen $\rightarrow$ (pk, sk) generates public and private PCF keys
- KeyDer(pk_A, sk_B) $\rightarrow K_B^{AB}$ yields Bob's PCF key w.r.t. Alice's key
- Eval$(K, x) \rightarrow y$ yields a pseudorandom sample

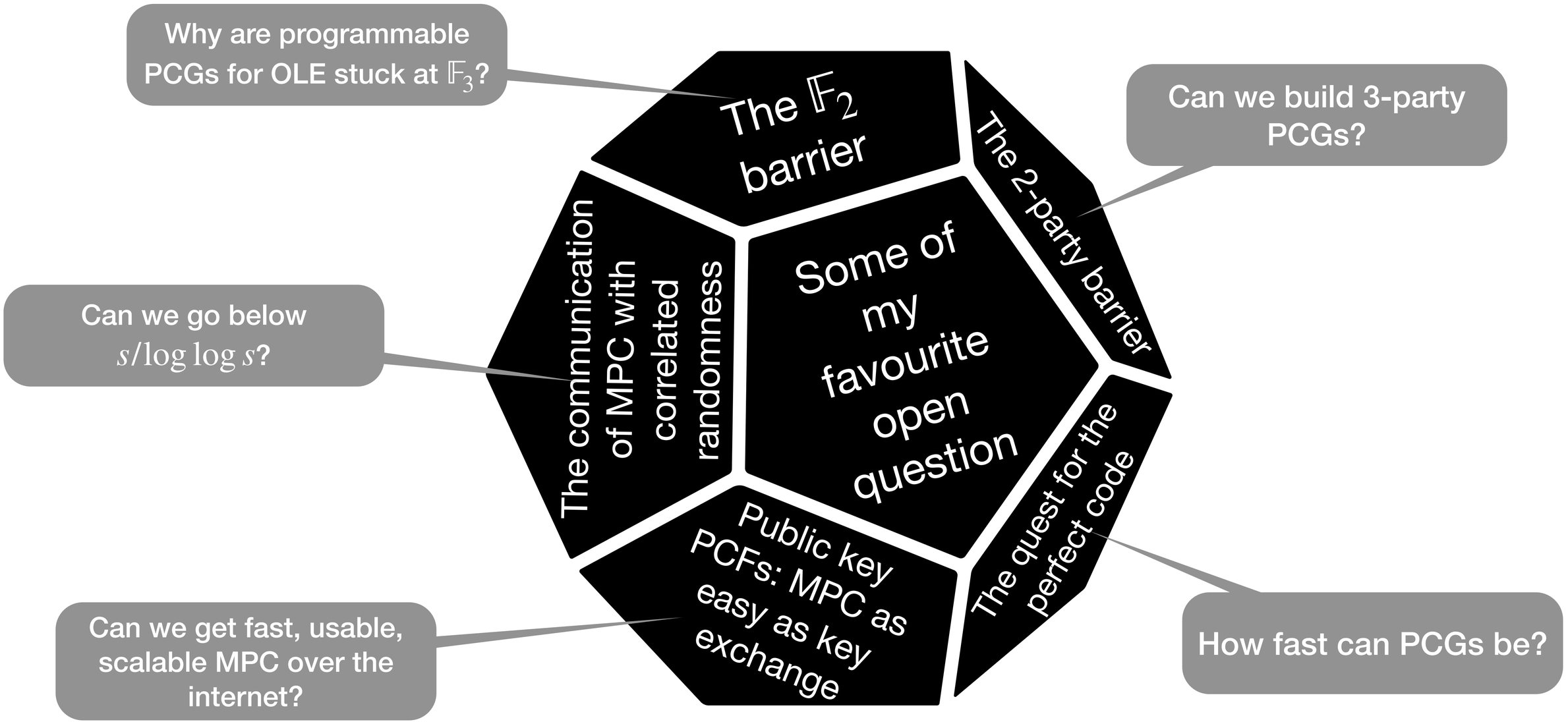
Public-Key Pseudorandom Correlation Functions

Public-key PCFs are exactly the *right tool* to enable scalable, on-demand 2-party secure computation over the Internet, with a communication and computation pattern close to that of secure *communication* over the web.

Building efficient public-key PCF is essentially a wide-open question: the recent work of EC:Orlandi-Scholl-Yakoubov'21 gets it for OT from QR, but efficiency is quite bad.

(Teaser) Coming soon: we have some exciting progress in this line of work, which does not fully solve the problem, but is a big step forward!

Some of my Favourite Open Questions



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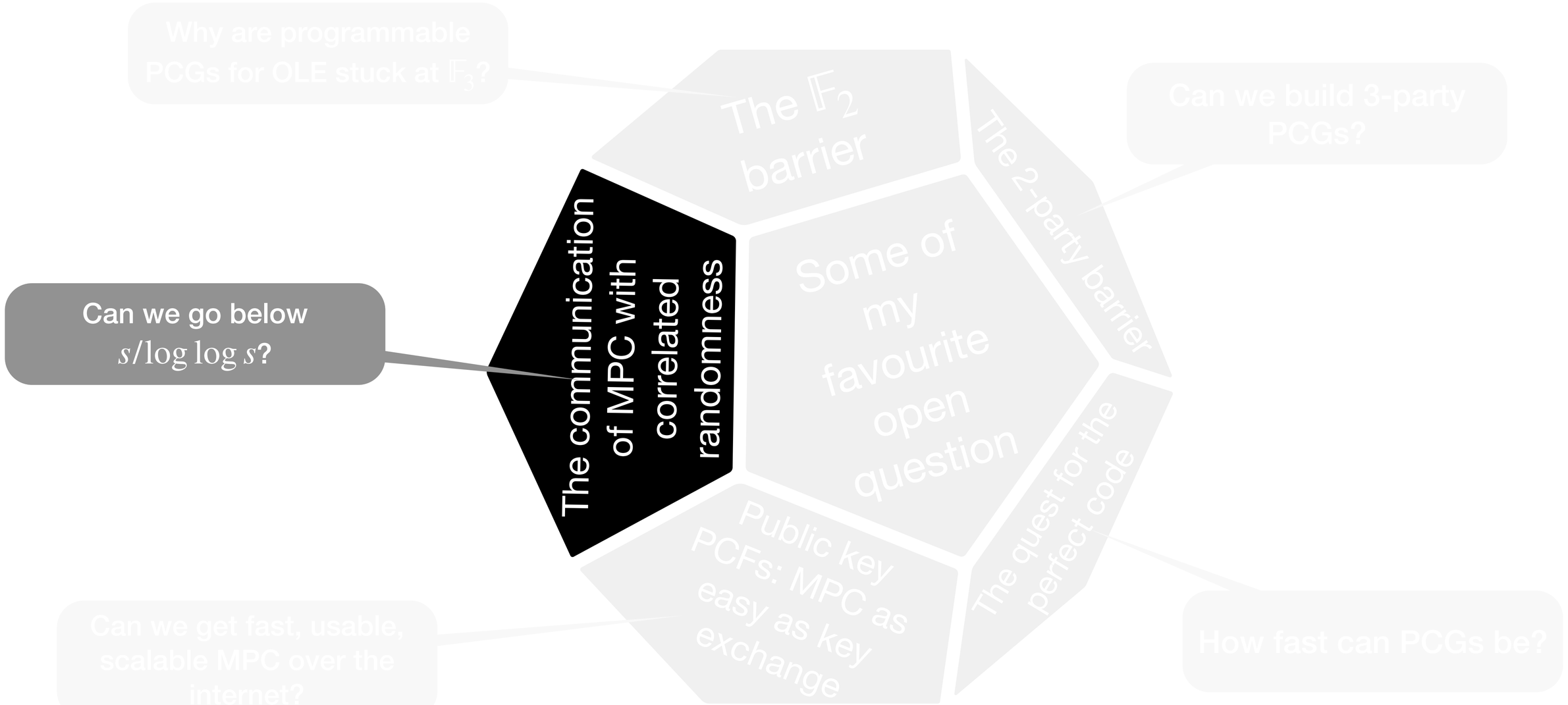

Some of my Favourite Open Questions

Can we go below $s/\log\log s$?

No time left for that, but I'd be happy to discuss it over dinner tonight!

Other cool things to check out that I don't have time to discuss:

- People have been doing great things in zero-knowledge using these PCG techniques (incl. right here in Aarhus!)
- Everything we have so far works only for two parties!
- … And many more

Questions?